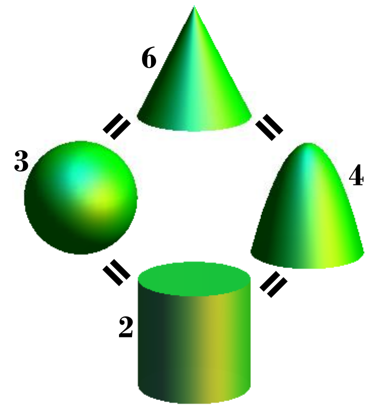
# 2012

## High School Math Contest




# Geometry Exam

## Lenoir-Rhyne University

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# Geometry Solutions

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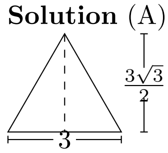
**GEOMETRY - Solutions, March, 2012**

1. **Solution** (C)
The circle of diameter 3 has area $\pi\left(\frac{3}{2}\right)^2$. The circle of diameter 1 has area $\pi\left(\frac{1}{2}\right)^2$. The ratio of the purple-painted area to the gold-painted area is $\dfrac{\pi\left(\frac{3}{2}\right)^2 - \pi\left(\frac{1}{2}\right)^2}{\pi\left(\frac{1}{2}\right)^2} = 8$.

2. **Solution** (B)
Let $r$ be the radius of the semicircular. Then $r^2 = (\sqrt{28})^2 + \left(\frac{\sqrt{28}}{2}\right)^2 = 28 + 7 = 35$. The area of the semicircle is $\frac{1}{2}\pi r^2 = \frac{35\pi}{2}$.

3. **Solution** (D)
From $x - 40° + 2x + 10° = 180°$ we get $x = 70°$. Then $30° = 3y - 18°$, which gives $y = 16°$. Therefore, $x + y = 70° + 16° = 86°$.

4. **Solution** (C)
The area of the square is $(2r)^2 = 4r^2$ and the area of the circle is $\pi r^2$. Hence, the shaded region has area $4r^2 - \pi r^2$.

5. **Solution** (A)


The area of the triangle is $\frac{1}{2}(3)\left(\frac{3\sqrt{3}}{2}\right) = \frac{9\sqrt{3}}{4}$ cm$^2$.

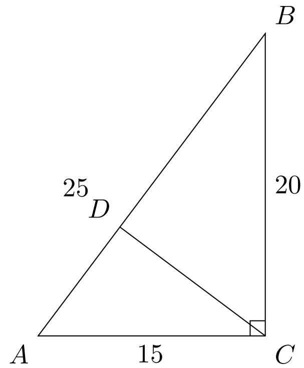
6. **Solution** (B)
The hight of the triangle is $h = \sqrt{10^2 - 6^2} = 8$ and its area is $\frac{1}{2}(12)(8) = 48$. Let $L$ and $W$ denote the length and the width of the rectangle respectively. Then $L = \frac{48}{W} = \frac{48}{4} = 12$. The perimeter of the rectangle is $2(12 + 4) = 32$.

7. **Solution** (B)
Let $A$ and $B$ be the centers of the circles. Then the Pythagorean theorem gives us that $PQ^2 = AB^2 - (AP - BQ)^2 = 25 - 9 = 16$. Hence $PQ = 4$.

8. **Solution** (D)
The surface area is given by $s^2 + 2sl$. Hence, $832 = 16^2 + 32l$. Therefore, $l = 18$ m.

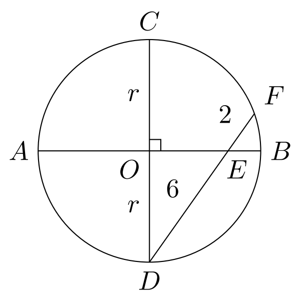
9. **Solution** (B)
The area is $\frac{1}{2}(b_1 + b_2) \cdot h = \frac{1}{2}(40 + 60) \cdot 25 = 1250$ ft$^2$. Since $\frac{1250}{15} = 83.\overline{3}$, the minimum number of bags of topsoil is 84.

10. **Solution** (B)
Since $15^2 + 20^2 = 25^2$, the triangle is a right triangle. The two legs $AC$ and $BC$ are altitudes with lengths 15 and 20 respectively. The area of $\triangle ABC$ is 150. From $\frac{25 \cdot DC}{2} = 150$, we get $DC = 12$.



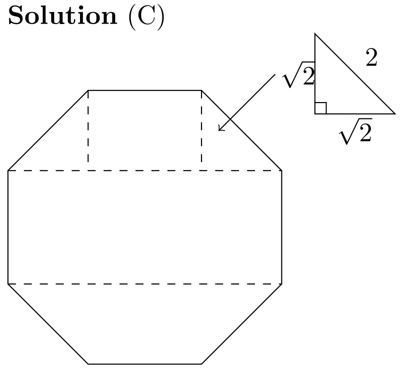

11. **Solution** (C)
The area of the region that consists of the equilateral triangle $AOB$ topped with the semicircle is $\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{8}\pi$. The area of the lune results from subtracting from this the area of the sector of the larger semicircle which is $\frac{\pi}{6}$. The area of the lune is $\frac{\sqrt{3}}{4} + \frac{1}{8}\pi - \frac{1}{6}\pi = \frac{1}{24}(6\sqrt{3} - \pi)$.

12. **Solution** (C)




$\triangle CFD$ is a right triangle becasue $CD$ is a diameter of the circle. Then $\triangle CFD \sim \triangle DOE$, so $\frac{2r}{6+2} = \frac{6}{r}$, i.e. $r^2 = 24$. The area of the circle is $24\pi$.

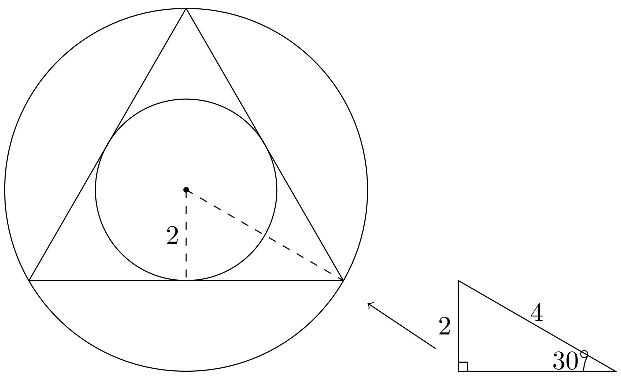
13. **Solution** (C)




Area $= 4\left(\frac{1}{2}(\sqrt{2})^2\right) + 2\left(2\sqrt{2}\right) + 2\left(2 + 2\sqrt{2}\right) = 4 + 4\sqrt{2} + 4 + 4\sqrt{2} = 8\left(1 + \sqrt{2}\right)$ ft$^2$

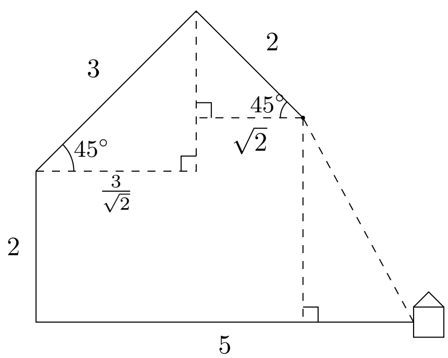
14. **Solution** (B)
Since $\angle CAT = 25^\circ = \angle EBF$, we have $\angle BFE = 180^\circ - 25^\circ - \angle FEB = 155^\circ - 95^\circ = 60^\circ$.

15. **Solution**
(B)



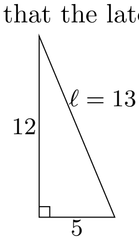

$\Rightarrow$ Diameter $= 8$

16. **Solution** (A)




$D = \sqrt{(2 + \frac{\sqrt{2}}{2})^2 + (5 - \frac{5\sqrt{2}}{2})^2} = \sqrt{42 - 23\sqrt{2}}$ ft.

17. **Solution** (D)
The lateral area is $\pi r \ell$ where $r$ is the radius and $\ell$ is the lateral surface length. Since $l = 13$, we get that the lateral area is $65\pi\text{cm}^2$.




18. **Solution** (A)
Of the 10 possible combinations only $(2, 3, 4)$, $(2, 4, 5)$, and $(3, 4, 5)$ satisfy the Triangle Inequality.

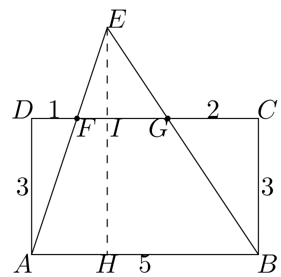
19. **Solution** (B)
The total area of the triangles $PAD$ and $PCB$ is half that of the parallelogram. Hence $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ of the area of the parallelogram is 6. The area of the parallelogram is 36.

20. **Solution** (C)
Denote the height by $h$. Then $7 \cdot 12 \cdot 15 = 7 \cdot 18 \cdot h$ or $h = \frac{12 \cdot 15}{18} = 10$ in.

21. **Solution** (C)
Drop perpendiculars $AH$ and $DK$ to $BC$ from $A$ and $D$, respectively. Then triangle $HAB$ is half a square with diagonal $\sqrt{6}$, so that $HA = HB = \sqrt{3}$. Also, triangle $KCD$ is half an equilateral triangle with side 6, so that $CK = 3$ and $DK = 3\sqrt{3}$. Hence $HK = 8$ while the difference of $AH$ and $DK$ is $2\sqrt{3}$. The length of $AD$ is $\sqrt{8^2 + (2\sqrt{3})^2} = 2\sqrt{19}$.

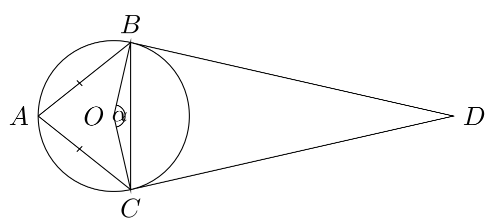
22. **Solution** (E)
Pythagorean Triple $(8, 15, 17)$ satisfies $17 - 8 = 9$. The area = of the triangle is $\frac{1}{2} \cdot 8 \cdot 15 = 60$.

23. **Solution** (D)




Let $H$ be on line $AB$ such that $EH \perp AB$, and let $I$ be the intersection of $EH$ and $DC$. Then $FG = 2$. Since $\triangle ABE \sim \triangle FGE$ we have $\frac{FG}{AB} = \frac{EI}{EH} = \frac{EI}{EI+IH}$. Hence $\frac{2}{5} = \frac{EI}{EI+3}$, i.e. $EI = 2$ and $EH = 5$. $A_{\triangle ABE} = \frac{1}{2} AB \cdot EH = \frac{25}{2}$.

24. **Solution** (A)




Let $\alpha$ be the central angle generated by the chord $BC$. Then $\angle DBC = \frac{\alpha}{2}$ (tangent chord theorem) and $\angle BAC = \frac{\alpha}{2}$ (inscribed angle theorem). Hence $\angle DBC = \angle BAC$. Both $\triangle BAC$ and $\triangle BDC$ are isosceles. Since $\angle ABC = 2\angle BDC$ we have

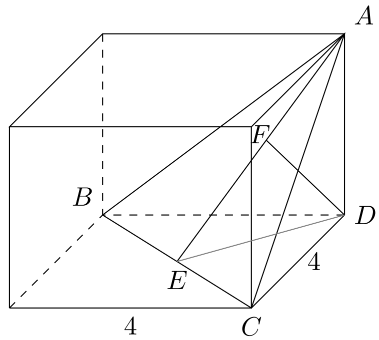
$$\pi = \angle BAC + \angle ABC + \angle ACB = \angle BAC + 4\angle BDC$$
$$\pi = \angle BDC + \angle DBC + \angle DCB = \angle BDC + 2\angle BAC$$

Hence $\angle BDC = \pi - 2\angle BAC$ and

$$\pi = \angle BAC + 4\angle BDC = \angle BAC + 4(\pi - 2\angle BAC) =$$
$$= 4\pi - 7\angle BAC$$

Therefore $\angle BAC = \frac{3\pi}{7}$.

25. **Solution** (A)




We have $CE = 2\sqrt{2}, \quad DE = 2\sqrt{2}$, $AE = \sqrt{AD^2 + DE^2} = \sqrt{17}$. Since $\triangle ADE$ is a right triangle, we have

$$A_{\triangle ADE} = \frac{2\sqrt{2} \cdot 3}{2} = 3\sqrt{2}$$
$$A_{\triangle ADE} = \frac{1}{2}\overline{AE} \cdot \overline{DF}$$
$$3\sqrt{2} = \frac{1}{2}\sqrt{17} \cdot \overline{DE} \Rightarrow \overline{DE} = \frac{6\sqrt{2}}{17} = \frac{6\sqrt{34}}{17}$$

26. **Solution** (C)
Half the diagonal of the square base has length $\frac{3\sqrt{2}}{2}$. Since the pyramid is half of a regular octahedron, its height has the same length. Hence the height of the top of the building from the ground is $2 + \frac{3\sqrt{2}}{2}$.

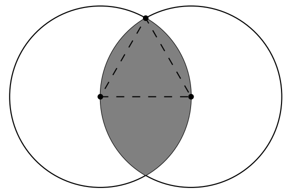
27. **Solution** (A)
The sum of the interior angles of an $n$-sided polygon is $(n-2) \cdot 180°$, so this $2n$-sided polygon has interior angle sum given by

$$(2n-2) \cdot 180° = \angle A_1 + \angle A_2 + \ldots + \angle A_n + (360° - \angle B_1) + \ldots + (360° - \angle B_n).$$

Since $\angle A_1 = \ldots = \angle A_n$, $\angle A_1 + 10° = \angle B_1 = \angle B_2 = \ldots = \angle B_n$, we have

$$2(n-1) \cdot 180° = n\angle A_1 + 360° n - n\angle A_1 - 10° n$$
$$360° n - 360° = 350° n$$
$$n = 36$$

28. **Solution** (D)



Sector area $= \frac{1}{6}\pi(4)^2 = \frac{8}{3}\pi$
Triangle area $= \frac{1}{2}(4)2\sqrt{3} = 4\sqrt{3}$
Shaded area $= 2\left(\frac{8}{3}\pi\right) + 2\left(\frac{8}{3}\pi - 4\sqrt{3}\right) = \frac{32}{3}\pi - 8\sqrt{3}$ cm$^2$.

29. **Solution** (D)
The diagonal crosses 32 horizontal grid lines and 74 vertical grid lines. However, the greatest common divisor of 33 and 75 is 3 so that it crosses exactly one corner of a square twice (at $\frac{1}{3}$ and $\frac{2}{3}$ distance). Thus it has $32 + 74 - 3 = 103$ points of intersections on it, and they divide it into 102 segments of positive length.

30. **Solution** (E)
The volume of a sphere is proportional to the cube of its radius. Hence $n \geq \frac{1000}{27}$. The minimum value of $n$ is 38 and it exceeds 20.