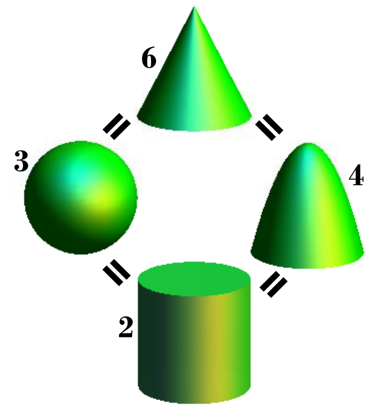

2012

High School Math Contest

Algebra II Exam

Lenoir-Rhyne University

Donald and Helen Schort School of Mathematics and Computing Sciences

Algebra II Solutions

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ALGEBRA II - Solutions, March, 2012

1. **Solution**
   (E)

   Consider the following: g(2) = 5 and f(5) = 10 Thus f(g(2)) = 10

2. **Solution**
   (A)

   The dimension-equation could be written as (3 x 2)*(a x b) = (3 x 2). To be able to multiply at all, a must be 2 and to obtain the matrix with two columns b must be 2 thus (2,2).

3. **Solution**
   (B)

   The equivalence equations for taking exponential form to logarithmic form are as follows:

   $$b^n = c$$

   $$\log_b c = n$$

   Thus, $y = 2^{1/4}$ can be rewritten as $\log_2 y = \frac{1}{4}$.

4. **Solution**
   (E)
   Using the properties of natural logarithms, observe that $e^{x^{2x+1}} = 1 \implies x^{2x+1} = \ln(1) = 0 \implies x = 0$. Note that solutions (a), (b), (c) all yield $x = 0$, so the answer is (E).

5. **Solution**
   (E)
   Observe that $\frac{1-x}{1-\frac{1}{1-\frac{1}{x}}} = \frac{1-x}{1-\frac{x}{x-1}} = \frac{1-x}{\frac{-1}{x-1}} = (1-x)^2$, so the answer is (E).

6. **Solution**
   (B)
   If x varies jointly as y and z then $x = kyz$, so we have that $x = k(2^{-y})(z^2)$ and using the given information $1 = k(2^{-1})(2^2) = 2k \implies k = \frac{1}{2}$, so $x = \frac{1}{2}(2^3)(\frac{1}{2})^2 = 1$, so the answer is (B).

7. **Solution**
   (D)
   -f(x) - 2 consists of an inversion of the original equation over the x-axis and a translation of 2 down the y-axis. Since all are graphically represented, there is little question as to what the answer is.

8. **Solution**
   (C)

   $g(2) = 2*(2)^2 + 2 = 10; f(10) = \frac{2*10+1}{-3} = \frac{21}{-3} = -7$

9. **Solution**
   (A)

$$\begin{aligned}\log_x(\frac{1}{9}) &= \frac{-2}{3}\\ x^(-2/3) &= \frac{1}{9}\\ \frac{1}{x^{2/3}} &= \frac{1}{9}\\ \frac{1}{x^{1/3}} &= \frac{1}{3}\\ \frac{1}{x} &= \frac{1}{27}\\ x &= 27\end{aligned}$$

10. **Solution**
    (E)

$$\begin{aligned}(x-(3+i))(x-(3-i)) &= (x-3-i)(x-3+i)\\ &= x^2-3x+xi-3x+9-3i-xi+3i-i^2\\ &= x^2-6x+9+1\\ &= x^2-6x+10\end{aligned}$$

11. **Solution**
    (D)
    By multiplying the denominator by the complex conjugate of $(1-10i)$, $(1+10i)$, we have $\frac{1+5i}{1-10i} = \frac{(1+5i)(1+10i)}{(1-10i)(1+10i)} = \frac{1+15i+50i^2}{1-100i^2} = \frac{15i-49}{101}$, so the answer is (D).

12. **Solution**
    (D)

    We obtain the equation for a circle as follows:

$$(x+xtranslation)^2+(y+ytranslation)^2 = radius^2$$
$$x^2+y^2 = 25$$
$$x^2+(4)^2 = 25$$
$$x = \pm 3$$

13. **Solution**
    (D)

    Solving the equations for y, the first three, and x, the final, we obtain the following:

$$y \leq 2*x+\frac{2}{3}$$

$$y \geq 2$$

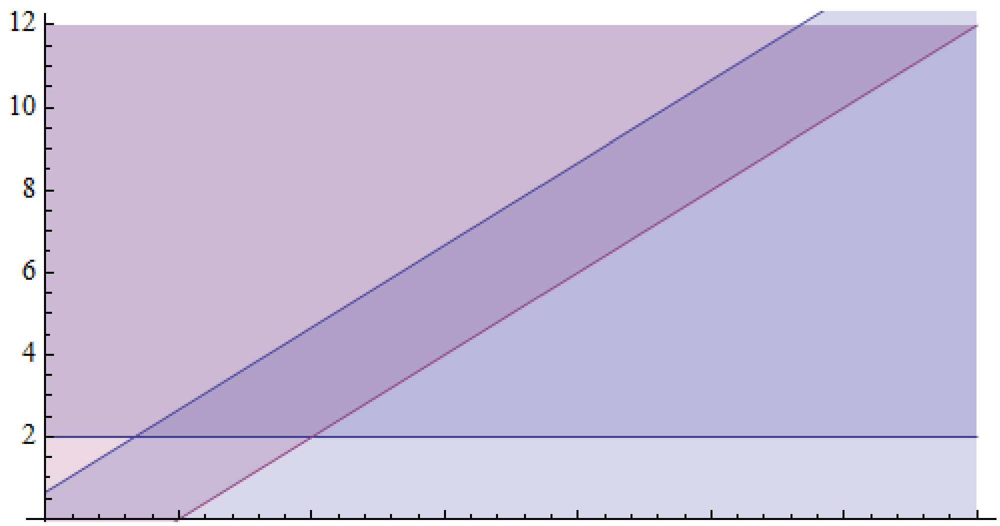
$$y \geq 2 * x - 2$$
$$x \leq 7$$

Which produces:(Mathematica)

```
p1 := Plot[{2 , 2 x - 2 }, {x, 0, 7}, Filling → Top]

p2 := Plot[{2 x + 2 / 3}, {x, 0, 7}, Filling → Bottom]

Show[p1, p2]
```



14. **Solution**
(B)

$$\begin{aligned} y &= \frac{\sqrt{6x+8}}{4} \\ x &= \frac{\sqrt{6y+8}}{4} \\ 4x &= \sqrt{6y+8} \\ 16x^2 &= 6y+8 \\ 16x^2 - 8 &= 6y \\ \frac{16x^2-8}{6} &= y \\ \frac{8x^2-4}{3} &= y \end{aligned}$$

15. **Solution**
(D)
Using the properties $\ln(a^b) = b\ln(a)$ and $\ln(ab) = \ln(a) + \ln(b)$, observe that $3^{6x} = 30 \implies 6x\ln(3) = \ln(30) \implies x = \frac{\ln(30)}{6\ln(3)} = \frac{\ln(3)+\ln(10)}{6\ln(3)} = \frac{1}{6}\left(1 + \frac{\ln 10}{\ln 3}\right)$, so the answer is (D).

16. **Solution**
(B)
Observe that

$$\begin{array}{r|l} & 2x^2 \quad + x \quad + 2 \\ \hline x-4 & 2x^3 - 7x^2 - 2x - 15 \\ & -2x^3 + 8x^2 \\ \hline & \qquad x^2 - 2x \\ & \qquad -x^2 + 4x \\ \hline & \qquad\qquad 2x - 15 \\ & \qquad\qquad -2x \quad +8 \\ \hline & \qquad\qquad\qquad -7 \end{array}$$

so the answer is $2x^2 + x + 2 - \frac{7}{x-4}$, which corresponds to (B).

17. **Solution**
(A)
First observe that $g(x) = x + 1 \implies g^{-1}(x) = x - 1 \implies g^{-1}(\frac{1}{x}) = \frac{1}{x} - 1 = \frac{1-x}{x}$. We then use $\frac{1-x}{x}$ in place of x in $f(x)$ and evaluate, where $f(g^{-1}(\frac{1}{x})) = f(\frac{1-x}{x}) = \frac{(1-x)^3}{x^3} - \frac{(1-x)^2}{x^2} = \frac{1-3x+3x^2-x^3-x+2x^2-x^3}{x^3} = \frac{1-4x+5x^2-2x^3}{x^3} = \frac{1}{x^3} - \frac{4}{x^2} + \frac{5}{x} - 2$, so the answer is (A).

18. **Solution**
(D)
Since $f(x)$ is undefined at $x = 1$ there is a vertical asymptote at $x = 1$. There is an x-intercept at $x = -1$ since $f(-1) = 0$ but not at $x = 1$ due to the vertical asymptote. The $y-$intercept is 4 since $f(0) = 4$. The answer is (D).

19. **Solution**
(E)
Both $f(x)$ and $g(x)$ have a horizontal asymptote at $y = 0$. The graph of $f(x)$ is undefined at $x = -2$ so a one-unit shift of $f(x)$ to the right yields the graph of $\frac{1}{x+1} \neq g(x)$. However, a one-unit shift of the graph of $g(x)$, which is undefined at $x = -3$ will transform it to $\frac{1}{x+2} = f(x)$. The answer is (E).

20. **Solution**
(D)

To more easily solve this problem, one should first translate any hours into minutes. It, then, takes Amanda 123 minutes and both still 41 minutes. One may notice that $123 = 41 * 3$. Thus, in 41 minutes Amanda can only paint 1/3rd of the room. This means that her roommate must complete 2/3rds of the job in 41 minutes and so $41 * 4/3 = 61.5$ minutes = 1 hour and 1&1/2 minutes.

21. **Solution**
(E)
The volume of the cylinder is $V = \pi r^2 h$ and $r = d/2 = \frac{1}{2}\frac{\sqrt{h}}{2} = \frac{\sqrt{h}}{4}$ so $r^2 = \frac{h}{16}$. Since $V = 4\pi = \pi r^2 h = \pi\frac{h^2}{16} \implies h^2 = 64 \implies h = 8$. The answer is (E).

22. **Solution**
(B)

$$\begin{aligned}\frac{2x+14}{x^3+7x^2-4x-28} &= \frac{2(x+7)}{(x^2-4)(x+7)}\\ &= \frac{2}{(x+2)(x-2)}\end{aligned}$$

Vertical asymptotes at $x = 2$ and $x = -2$

23. **Solution**
(D)

$$\begin{aligned}\log_3\left(\frac{x^2+7x+6}{x^2-4x-12}\right) - \log_3\left(\frac{x^2+19x+78}{x^2+7x+12}\right)\\ \log_3\left(\frac{(x+6)(x+1)}{(x-6)(x+4)}\right) - \log_3\left(\frac{(x+6)(x+13)}{(x+3)(x+4)}\right)\\ \log_3\left(\frac{(x+6)(x+1)}{(x-6)(x+4)} \times \frac{(x+3)(x+4)}{(x+6)(x+13)}\right)\\ \log_3\left(\frac{(x+1)}{(x-6)} \times \frac{(x+3)}{(x+13)}\right)\\ \log_3\left(\frac{(x+1)(x+3)}{(x-6)(x+13)}\right)\\ \log_3\left(\frac{x^2+4x+3}{x^2+7x-78}\right)\end{aligned}$$

24. **Solution**
(A)
Note: $x = -5$ and $x = 1$ cannot be part of the solution as the cause the denominator of the fraction to equal zero! $x = -4$, $x = -3$ and $x = 0$ all cause the numerator to equal zero which causes the entire fraction to equal zero and are thus part of the solution.

$$(-\infty, -5) \cup [-4, -3] \cup 0 \cup (1, \infty)$$

25. **Solution**
(C)

$$\begin{aligned}(1) &\quad 8H + 6F + 6C = 26.10\\ (2) &\quad 10H + 6F + 8C = 31.60\\ (3) &\quad 3H + 2F + 4C = 10.95\end{aligned}$$

Combine equations -(1) and (2) to get equation (4): $2H + 2C = 5.5$

Combine equations (1) and -3(3) to get equation (5): $-H - 6C = -6.75$

Combine Equations (4) and 2(5) to get equation (6): $-10C = -8$

Solve for C: $C = .80$

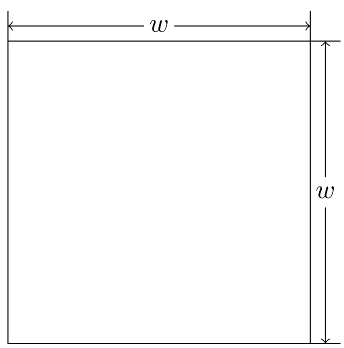
Plug .80 in for C in equation (4) and solve for H, which leads to $H = 1.95$

2 deluxe hamburgers plus 1 large cola is \$4.70

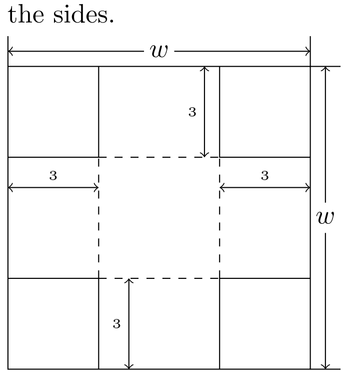
26. **Solution**
(B)
$3x^2 - 18x + 20$ has a vertex at (3,-7) and $2x^2 + 24x + 77$ has a vertex at (-6,5):

$$\begin{aligned} d &= \sqrt{(-6-3)^2 + (5-(-7))^2} \\ &= \sqrt{(9)^2 + (12)^2} \\ &= \sqrt{81 + 144} \\ &= \sqrt{225} \\ &= 15 \end{aligned}$$

27. **Solution**
(B)
Solution: Here is a picture. I don't know how big the cardboard will be yet, so I'll label the sides as having length "w".

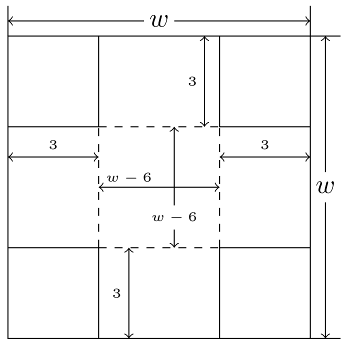


Since I know I'll be cutting out three-by-three squares to get sides that are three inches high, I can mark that on my drawing. The dashed lines show where I'll be scoring the cardboard and folding up the sides.



Since I'll be losing three inches on either end of the cardboard when I fold up the sides, the final width of the bottom will be the original "w" inches, less three on the one side and another three on the other side. That is, the width of the bottom will be $w - 3 - 3 = w - 6$. Then the volume of the box, from the drawing, is:

$$(w-6)(w-6)(3) = 48$$
$$(w-6)(w-6) = 16$$
$$(w-6)^2 = 16$$



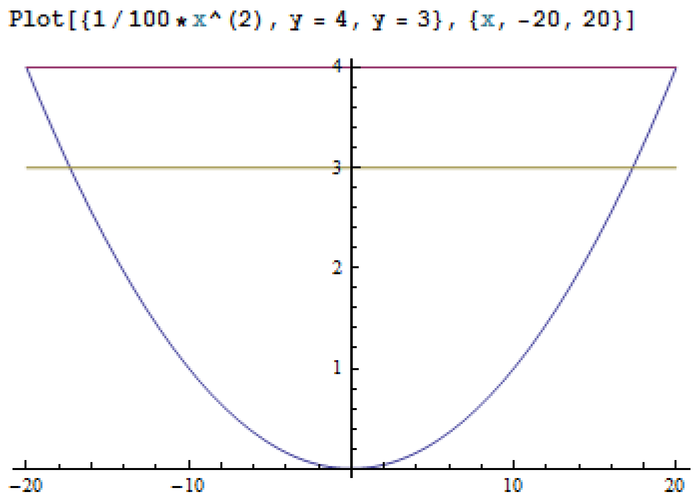
This is the quadratic I need to solve. I can take the square root of either side, and then add the to the right-hand side: $w = 6 \pm \sqrt{16} = 2$ or 10 in How do I know which solution value for the width is right? By checking each value in the original word problem. If the cardboard is only 2 inches wide, then how on earth would I be able to fold up three-inch-deep sides? But if the cardboard is 10 inches, then I can fold up three inches of cardboard on either side, and still be left with 4 inches in the middle. Checking: $(4)(4)(3) = 48$

28. **Solution**
    (E)

    We first should determine the equation for the parabola. Setting the minimum of the parabola at the origin, we find that the horizontal line y = 4 will represent the average height of the river graphically. The normal parabola $y = x^2$ is only 40 meters wide (20 + 20) at y = 4 so we need to reduce its y-values by a tenth, creating $y = (1/10 * x)^2 = 1/100 * x^2$. We want to find at what distance from the bank to indicate a depth of three meters. Therefore

$$3 = 1/100 * x^2$$
$$300 = x^2$$
$$10 * (3)^{1/2} = x$$

    The distance form the bank to the point where the depth of the river is 3m is $20 - 10\sqrt{3}$m.



(Generated in Mathematica)

29. **Solution**
(E)

$$r = 1 - (\frac{x}{soldprice})^{\frac{1}{n}}$$

$$0.065 = 1 - \frac{x}{32000}^{\frac{1}{9}}$$

$$\frac{x}{32000} = (1 - 0.065)^9$$

$$x = 32000 * (1 - 0.065)^9$$

30. **Solution**
(E)
Let d denote the rate of depreciation where $d = \frac{i}{100}$, thus the house depreciates in value by a factor of $1 - d$ during the first year and by a factor of $1 - 2d$ during the second year. Now observe that $100,000(1 - d)(1 - 2d) = 37,500 \implies (1 - d)(1 - 2d) = .375 \implies 1 - 3d + 2d^2 = .375 \implies 2d^2 - 3d + .625 = 0 \implies d = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(.625)}}{2(2)} = \frac{3 \pm 2}{4} = .25$ so $i = (.25)(100) = 25$, so the answer is (E).