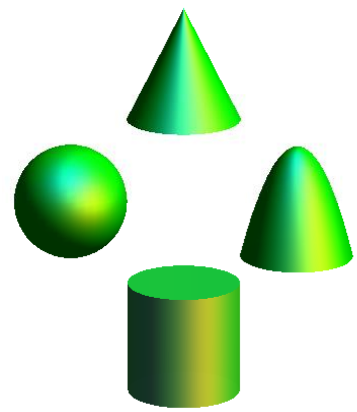
2015
High School Math Contest

Level 3 Exam Key



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1. A quadratic equation $y = ax^2 + bx + c$ is known to pass through the points $(-1, 5)$, $(0, -1)$, and $(3, 5)$. Find the sum of the coefficients.

   (A) -1 (B) 2 (C) 9 *(D)* -3 (E) None of the answers (A)–(D) is correct.

   Solution: We automatically know that $c = -1$ since the parabola passes through $(0, -1)$. To find a and b, we note that

   $$\begin{aligned} a - b &= 6, \\ 9a + 3b &= 6. \end{aligned}$$

   The first equation produces $a = b+6$. Substituting into the second equation, we have $9b+54+3b = 6$, so $12b = -48$. Then $b = -4$ and $a = 2$. The sum of the coefficients is $a + b + c = 2 - 4 - 1 = -3$.

2. If $x^2 + 2Ax + B = (x - A)(x - B)$ for all numbers x, and $A \neq B$, then B is equal to

   (A) -3 (B) 0 (C) 1 (D) 3 (E) None of the answers (A)–(D) is correct.

   Solution: Since $(x - A)(x - B) = x^2 + (-A - B)x + AB$, by equating coefficients we must have $2A = -A - B$ and $B = AB$. The only solution to these equations with $A \neq B$ is $A = 1$ and $B = -3$.

3. Simplify the expression $\sqrt{\dfrac{\sqrt{5}-2}{\sqrt{5}+2}}$.

   (A) $\dfrac{1}{\sqrt{5}-\sqrt{2}}$ (B) $\dfrac{1}{\sqrt{5}-2}$ *(C)* $\sqrt{5}-2$ (D) $\sqrt{5}+2$

   (E) None of the answers (A)–(D) is correct.

   Solution: First we simplify the fraction inside the square root by multiplying by the conjugate.

   $$\frac{\sqrt{5}-2}{\sqrt{5}+2} \cdot \frac{\sqrt{5}-2}{\sqrt{5}-2} = \frac{1}{\left(\sqrt{5}-2\right)^2}.$$

   Then we take the square root of this result, and simplify again by multiplying by the conjugate.

   $$\sqrt{\frac{1}{\left(\sqrt{5}-2\right)^2}} = \frac{1}{\sqrt{5}-2} = \frac{1}{\sqrt{5}-2} \cdot \frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{\sqrt{5}+2}{1} = \sqrt{5}+2.$$

4. If $2x + 5y = 3$ and $4x^2 - 25y^2 = 24$, find the value of $x - \frac{5}{2}y$.

(A) 2 *(B) 4* (C) 8 (D) 16 (E) None of the answers (A)–(D) is correct.

Solution: This problem can be solved by substitution, but there is another way. The second equation can be written as $(2x+5y)(2x-5y) = 24$. Since the first equation states that $2x+5y = 3$, a substitution into the second equation produces $2x - 5y = 8$. If we divide this equation by 2, we get that $x - \frac{5}{2}y = 4$.

5. If $(\sqrt[3]{2})^{2-x} = 2^{x^2}$, then x has two solutions a and b. What is $a + b$?

(A) $-\dfrac{2}{3}$ *(B)* $-\dfrac{1}{3}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{2}{3}$ (E) None of the answers (A)–(D) is correct.

Solution: The left-hand side is $\sqrt[3]{2}^{2-x} = (2^{1/3})^{2-x} = 2^{(2-x)/3}$, so we need $\frac{2-x}{3} = x^2$, or equivalently, $3x^2+x-2 = 0$. This factors to $(3x-2)(x+1) = 0$, so that $a = \frac{2}{3}$ and $b = -1$. Hence $a+b = \frac{2}{3}-1 = \frac{1}{3}$.

6. Find the equation of the line perpendicular to the line $2x + 4y = -3$ that passes through the center of the circle $x^2 + 6x + y^2 - 8y = 36$.

(A) $y = -2x - 2$ (B) $y = -2x + 2$ (C) $y = 2x - 10$ *(D)* $y = 2x + 10$

(E) None of the answers (A)–(D) is correct.

Solution: Completing the square, the circle has equation $(x + 3)^2 + (y - 4)^2 = 61$, so its center is $(-3, 4)$. The given line has equivalent equation $y = -\frac{1}{2}x - \frac{3}{4}$. The perpendicular slope is $m = 2$. The line containing $(-3, 4)$ with slope $m = 2$ has equation:

$$\begin{aligned} y - 4 &= 2(x + 3) \\ y &= 2x + 10 \end{aligned}$$

7. Find the general expression of a quadratic function whose x-intercepts are $(-4, 0)$ and $(2, 0)$ and whose range is $y \geq -18$.

(A) $f(x) = x^2+2x-8$ (B) $f(x) = x^2-2x+8$ *(C)* $f(x) = 2x^2+4x-16$ (D) $f(x) = 2x^2-4x+16$

(E) None of the answers (A)–(D) is correct.

Solution: Since we are given two x-intercepts, we know that the parabola will have the form $f(x) = C \cdot (x+4)(x-2)$ for some C. To get the range $y \geq -18$, the parabola will need to open up, so $C > 0$. The x-coordinate of the vertex will be halfway between the x-intercepts, at $x = -1$. For the range of the parabola to be $y \geq -18$, the vertex must be at $(-1, -18$. So

$$-18 = f(-1) = C(3)(-3) = -9C.$$

Hence $-9C = -18$, so $C = 2$. Therefore $f(x) = 2(x+4)(x-2) = 2x^2 + 4x - 16$.

8. Mary has three marbles of different colors, but the same size, in a bag. She randomly chooses a marble from the bag, looks at it, and puts it back. She repeats the process two more times. What is the probability that she has seen at most two different marbles?

   (A) 7/9 (B) 1/3 (C) 2/9 (D) 15/16 (E) None of the answers (A)–(D) is correct.

Solution: The number of ways to choose three marbles in this way, ignoring color, is $3 \cdot 3 \cdot 3 = 27$. The number of ways to choose three marbles of different colors is $3 \cdot 2 \cdot 1 = 6$. Hence the probability of choosing three marbles of different colors is $\frac{6}{27} = \frac{2}{9}$. By the complement, the probability of choosing three marbles with at least one color repeated is

$$1 - \frac{2}{9} = \frac{7}{9}.$$

9. If $f(x) = 2^x$ then 4^8 is equal to:

   (A) $f(f(2))$ *(B) $f(f(f(2)))$* (C) $f(f(f(f(2))))$ (D) $f(f(f(f(f(2)))))$

   (E) None of the answers (A)–(D) is correct.

Solution: We have

$$4^8 = (2^2)^8 = 2^{2 \cdot 8} = 2^{16} = 2^{(2^4)} = 2^{\left(2^{(2^2)}\right)} = f(f(f(2))).$$

10. If $\dfrac{(2+i)(3+2i) - (1+2i)(x+yi)}{2+i} = 1$, calculate $\dfrac{x}{y}$.

    (A) 9 (B) 14 (C) 6 *(D) 7* (E) None of the answers (A)–(D) is correct.

Solution: We need the top of the fraction to equal $2+i$.

$$\begin{aligned}(2+i)(3+2i)-(1+2i)(x+yi) &= (6+3i+4i+2i^2)-(x+2xi+yi+2yi^2)\\ &= 6+7i-2-x-2xi-yi+2y\\ &= (4-x+2y)+(7-2x-y)i.\end{aligned}$$

Therefore we need to:

$$\begin{aligned}-x+2y &= -2\\ 2x+y &= 6\end{aligned}$$

Solving this system, we have $x=\frac{14}{5}$ and $y=\frac{2}{5}$. Their ratio is $(\frac{14}{5})/(\frac{2}{5})=\frac{14}{2}=7$.

11. The equation $8^{x^2+3x}=32^{-3x+3}$ has two solutions a and b. What is $a\cdot b$?

(A) -3 (B) -4 *(C)* -5 (D) -6 (E) None of the answers (A)–(D) is correct.

Solution: Finding a common base, $2^{3x^2+9x}=2^{-15x+15}$. Therefore we need $3x^2+9x=-15x+15$, or equivalently, $3x^2+24x-15=0$. This can be written as $3(x^2+8x-5)=0$, so that a and b are the roots of $x^2+8x-5=0$. Since this would factor into $(x-a)(x-b)$, we automatically have that $a\cdot b=-5$. Instead, we could instead use the quadratic formula,

$$x=\frac{-8\pm\sqrt{64+20}}{2}=\frac{-8\pm\sqrt{84}}{2}=-4\pm\sqrt{21}$$

so that $a\cdot b=(-4+\sqrt{21})(-4-\sqrt{21})=16-21=-5$.

12. Solve for x: $\log_8(x+3)=\log_{16}(81)$.

(A) 24 (B) 75/2 (C) 6 (D) 78 (E) None of the answers (A)–(D) is correct.

Solution: Using the change of base formula, we have

$$\begin{aligned}\frac{\log_2(x+3)}{\log_2(8)} &= \frac{\log_2(81)}{\log_2(16)},\\ \frac{1}{3}\log_2(x+3) &= \frac{1}{4}\log_2(81),\\ \frac{1}{3}\log_2(x+3) &= \log_2(81^{1/4}),\\ \log_2(x+3) &= 3\cdot\log_2(3),\\ \log_2(x+3) &= \log_2(3^3)=\log_2(27),\end{aligned}$$

and therefore $x+3=27$, so that $x=24$.

13. Suppose a function $f(x)$ satisfies $f(x) + 4f(3 - x) = 5x$ for all real numbers x. Then $f(3)$ is

(A) -3 (B) -2 (C) 0 (D) 4 *(E) None of the answers (A)–(D) is correct.*

Solution: Plugging in $x = 0$ and $x = 3$, we obtain:

$$f(0) + 4f(3) = 0$$
$$f(3) + 4f(0) = 15$$

Solving this system gives $f(3) = -1$.

14. Which of the following are true about the transformation of $f(x)$ to $g(x)$ if $f(x) = x^2$ and $g(x) = 3x^2 - x + 4$?

(A) Shifted up by $\dfrac{1}{12}$. *(B) Shifted right by $\dfrac{1}{6}$.* (C) Shifted left by $\dfrac{1}{6}$.

(D) Both (A) and (B) are correct. (E) Both (A) and (C) are correct.

Solution: By completing the square we have $g(x) = 3(x - \frac{1}{6})^2 + \frac{47}{12}$. It is shifted up by $\frac{47}{12}$ and right by $\frac{1}{6}$.

15. One of Jason and Jerry lies on Mondays, Tuesdays and Wednesdays, and tells the truth on the other days of the week. The other lies on Thursdays, Fridays and Saturdays, and tells the truth the other days of the week. At noon, the two had the following conversation:

Jerry: I lie on Saturdays.

Jason: I will lie tomorrow.

Jerry: I lie on Sundays.

This conversation takes place on a:

(A) Monday (B) Tuesday *(C) Wednesday* (D) Friday (E) Saturday

Solution: Jerry must be lying because he can't lie on Sunday. So he must tell the truth on Saturdays. The conversation then takes place on Monday, Tuesday or Wednesday. It must be Wednesday because that is the only way Jason's statement can be true.

16. A square is inscribed within a circle. If the ratio of circumference to area of the circle is 7, what is the ratio of perimeter to area for the square?

(A) $7\sqrt{2}$ (B) $2\sqrt{7}$ (C) $\frac{7}{\sqrt{2}}$ (D) $\frac{2}{\sqrt{7}}$ (E) None of the answers (A)–(D) is correct.

Solution: Let r be the radius of the circle. Then its circumference is $2\pi r$ and its area is πr^2, so the ratio of circumference to area is $\frac{2\pi r}{\pi r^2} = \frac{2}{r}$. Hence $\frac{2}{r} = 7$, so $r = \frac{2}{7}$.

Let x be the length of a side of the square. The diagonal of the square is a diameter of the circle, so its length is $2r$. Therefore half of the square is a right triangle with hypotenuse length $2r$ and leg lengths both x. By the Pythagorean Theorem we have $(2r)^2 = x^2 + x^2$, so $4r^2 = 2x^2$, so

$$x = \sqrt{2} \cdot r = \frac{2\sqrt{2}}{7}.$$

Then the ratio of perimeter to area for the square is

$$\frac{4x}{x^2} = \frac{4}{x} = \frac{4}{2\sqrt{2}/7} = \frac{14}{\sqrt{2}} = \frac{14\sqrt{2}}{2} = 7\sqrt{2}.$$

17. I want to triple the volume in a cylindrical can. Due to package limitations, I can only increase the height of the can by 50%. Assuming I do this, approximate the percentage that the radius of the can must be increased to triple the volume.

(A) 41.4% (B) 45.2% (C) 52.1% (D) 53.8% (E) None of the answers (A)–(D) is correct.

Solution: Let r and h be the radius and height of the original cyclinder, respectively. Then its volume is $V = \pi r^2 h$. Let R be a new radius such that the volume of the new cylinder is $3V = \pi R^2(1.5h)$. Then

$$\frac{3V}{V} = \frac{\pi R^2(1.5h)}{\pi r^2 h},$$

which reduces to $\frac{R}{r} = \sqrt{2} \approx 1.414$, so the radius must be increased by approximately 141.4%.

18. Let O be the center of a circular pond of radius 7 meters. The point A is on the edge of the pond, due east of O. A boy swims westward from A to the point B, 3 meters from A. Then he turns north and swims to shore, whereupon he swims westward again until he reaches the point C due north of O. The distance in meters between O and C is:

(A) $2\sqrt{7}$ (B) 4 (C) $7\sqrt{3}$ *(D)* $\sqrt{33}$ (E) None of the answers (A)–(D) is correct.

Solution: Let D be the point on the shore that the boy reaches before swimming to point C. Observe that the distance from B to D is the same as the distance from O to C.

Since the distance from O to A is 7 meters, and the distance from A to B is 3 meters, we know the distance from O to B is $7 - 3 = 4$ meters. The distance from O to D is the radius of the circle, which is 7. Then O, B, and D are the corners of a right triangle with hypotenuse length 7 and one leg length 4. By the Pythogorean Theorem, the other leg has length

$$\sqrt{7^2 - 4^2} = \sqrt{49 - 16} = \sqrt{33}.$$

This is the distance from B to D, which means it is also the distance from O to C.

19. Which of the following angles does not share its terminal side with the others?

(A) $-\dfrac{310\pi}{45}$ (B) $1640°$ (C) $-3760°$ (D) $-\dfrac{62\pi}{9}$

(E) None of the answers (A)–(D) is correct.

Solution: All of the angles are coterminal:

$$-\frac{310\pi}{45} + 4(2\pi) = -\frac{310\pi}{45} + 4\left(\frac{90\pi}{45}\right) = \frac{10\pi}{9} = 200°$$

$$1640° - 4(360° = 200°$$

$$-3760° + 11(360° = 200°$$

$$-\frac{62\pi}{9} + 4(2\pi) = -\frac{62\pi}{9} + 4\left(\frac{18\pi}{9}\right) = \frac{10\pi}{9} = 200°$$

So the answer is "None of the answers (A)–(D) is correct."

20. You drive a car with 20-inch diameter tires and are traveling on the turnpike at a constant speed of 80 mph. If you have been traveling under these conditions for 30 minutes, how many complete revolutions did your tires make?

(A) $\dfrac{173200}{\pi}$ (B) $\dfrac{153190}{\pi}$ (C) $\dfrac{146430}{\pi}$ (D) $\dfrac{132380}{\pi}$

(E) None of the answers (A)–(D) is correct.

Solution: You have driven 40 miles. There are 5280 feet in 1 mile, and 12 inches in 1 foot, so you have driven 2,534,400 inches. The circumference of the tire is 20π inches. Therefore, one complete revolution occurs every 20π inches. After 40 miles, the tire has made

$$\frac{2,534,400}{20\pi} = \frac{126,720}{\pi}$$

revolutions. Therefore, the answer is "None of the answers (A)–(D) is correct."

21. A survey of students taking both college algebra and statistics at a university found that 70% received a passing grade in college algebra, 75% received a passing grade in statistics, and 60% passed both courses. If a person received a passing grade in statistics, what is the probability that the person also passed college algebra?

(A) 60% (B) 70% *(C)* 80% (D) 90% (E) None of the answers (A)–(D) is correct.

Solution: This is a conditional probability problem. If we let C be the event that a person passes college algebra and S be the event that a person passes statistics, then $P(C \cap S) = P(C|S) \cdot P(S)$, or equivalently, $P(C|S) = \dfrac{P(C \cap S)}{P(S)}$. We have that $P(C \cap S) = \frac{60}{100} = \frac{3}{5}$ and $P(S) = \frac{75}{100} = \frac{3}{4}$. Therefore

$$P(C|S) = \frac{3/5}{3/4} = \frac{3}{5} \cdot \frac{4}{3} = \frac{4}{5} = 80\%.$$

22. Find the solution of the inequality $2 - \Big| x - |5 - 4x| + 3 \Big| \leq 0$.

(A) $(-\infty, 0] \cup [\frac{2}{3}, 4] \cup [\frac{15}{2}, \infty)$ *(B)* $(-\infty, 0] \cup [\frac{4}{5}, 2] \cup [\frac{10}{3}, \infty)$ (C) $(-\infty, 0] \cup [\frac{3}{2}, 3] \cup [\frac{11}{3}, \infty)$

(D) $(-\infty, 0] \cup [\frac{5}{4}, 4] \cup [\frac{16}{3}, \infty)$ (E) None of the answers (A)–(D) is correct.

Solution: $2 - \Big| x - |5 - 4x| + 3 \Big| \leq 0$ is equivalent to $\Big| x - |5 - 4x| + 3 \Big| \geq 2$, so that either

$$x - |5 - 4x| + 3 \geq 2 \qquad \text{OR} \qquad x - |5 - 4x| + 3 \leq -2.$$

- Simplifying the left inequality, we have $|5 - 4x| \leq x + 1$. Hence $-(x + 1) \leq 5 - 4x \leq x + 1$, so $-x - 1 \leq 5 - 4x$ and $5 - 4x \leq x + 1$. Observe that

$$\begin{aligned} -x - 1 \leq 5 - 4x &\implies 3x \leq 6 \implies x \leq 2, \\ 5 - 4x \leq x + 1 &\implies 4 \leq 5x \implies \tfrac{4}{5} \leq x. \end{aligned}$$

  Therefore $\frac{4}{5} \leq x \leq 2$.

- Simplifying the right inequality, we have $x+5 \le |5-4x|$. Hence $5-4x \ge x+5$ or $5-4x \le -(x+5)$. Observe that

$$\begin{aligned} 5-4x \ge x+5 &\implies 0 \ge 5x &&\implies 0 \ge x, \\ 5-4x \le -x-5 &\implies 10 \le 3x &&\implies \tfrac{10}{3} \le x. \end{aligned}$$

  Therefore $x \le 0$ or $x \ge \frac{10}{3}$.

Putting these two parts together, we conclude that $x \le 0$ or $\frac{4}{5} \le x \le 2$ or $\frac{10}{3} \le x$, so the solution set of the inequality is

$$(-\infty, 0] \cup [\tfrac{4}{5}, 2] \cup [\tfrac{10}{3}, \infty).$$

23. Let N be the sum of all prime numbers which divide 2015. Then the number of distinct prime numbers which divide N is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) None of the answers (A)–(D) is correct.

Solution: Since $2015 = 5 \cdot 13 \cdot 31$, the sum of its prime factors is $N = 5 + 13 + 31 = 49$. Because $N = 49 = 7^2$, there is only one prime number that divides N.

24. Simplify the following:

$$\left(\log_{128} 1331\right)\left(\log_{729} 1024\right)\left(\log_{3125} 81\right)\left(\log_{121}(5^{-1})\right)$$

(A) $-\frac{15}{28}$ *(B)* $-\frac{2}{7}$ (C) $-\frac{5}{8}$ (D) $-\frac{5}{14}$ (E) None of the answers (A)–(D) is correct.

Solution: First we use the natural logarithm to separate the logarithm bases from the logarithm inputs, and then we rearrange things and switch to conveniently-based logarithms:

$$
\begin{aligned}
&\big(\log_{128} 1331\big)\big(\log_{729} 1024\big)\big(\log_{3125} 81\big)\big(\log_{121}(5^{-1})\big) \\
&= \left(\frac{\ln 1331}{\ln 128}\right)\left(\frac{\ln 1024}{\ln 729}\right)\left(\frac{\ln 81}{\ln 3125}\right)\left(\frac{\ln(5^{-1})}{\ln 121}\right) \\
&= \left(\frac{\ln 1331}{\ln 121}\right)\left(\frac{\ln 1024}{\ln 128}\right)\left(\frac{\ln 81}{\ln 729}\right)\left(\frac{\ln(5^{-1})}{\ln 3125}\right) \\
&= \left(\frac{\log_{11} 1331}{\log_{11} 121}\right)\left(\frac{\log_2 1024}{\log_2 128}\right)\left(\frac{\log_9 81}{\log_9 729}\right)\left(\frac{\log_5(5^{-1})}{\log_5 3125}\right) \\
&= \left(\frac{3}{2}\right)\left(\frac{10}{7}\right)\left(\frac{2}{3}\right)\left(\frac{-1}{5}\right) \\
&= -\frac{2}{7}
\end{aligned}
$$

25. Town A and Town B are 225 miles apart. Car A leaves Town A at noon, and drives toward Town B at 60 mph. At 1pm, Car B leaves Town B at 45 mph, driving along the same road toward Town A. The speed limit for the road is set at 50mph. After driving 20 miles, Car A is pulled over by the police and cited for speeding. After an additional 22 minutes, he is given a ticket and can continue driving toward Town B, albeit driving a constant 50 mph. At what time do the cars meet?

(A) 2:40 pm (B) 2:45 pm (C) 2:50 pm *(D) 3:00 pm*

(E) None of the answers (A)–(D) is correct.

Solution: This is a twist on the "standard" car problem. Car A leaves first, driving 60 mph. The driver is pulled over after 20 miles, which at 60 mph took 20 minutes, so the time is 12:20 pm. Car A's driver starts driving again 22 minutes later, at 12:42 pm, driving 50 mph. In 18 minutes until 1:00 pm, the driver covers an additional

$$\left(\frac{50\,\text{miles}}{1\,\text{hour}}\right)\left(\frac{1\,\text{hour}}{60\,\text{minutes}}\right)(18\,\text{minutes}) = 15\,\text{miles}.$$

So Car A has driven 35 miles down the road at 1 pm. Therefore, this reduces to a "standard" car problem, where the cars can be considered to be 190 miles apart, driving toward each other starting at 1:00 pm. If they meet t hours after 1:00 pm, then:

$$
\begin{aligned}
50t + 45t &= 190 \\
95t &= 190 \\
t &= \frac{190}{95} = 2
\end{aligned}
$$

Hence they meet 2 hours after 1:00 pm, which is 3:00 pm.

26. Recall that all consecutive elements of an arithmetic sequence satisfy $a_{n+1} = a_n + d$ for some constant d. Suppose that the sum of the first and fourth terms of an increasing arithmetic sequence is 2, and the sum of their squares is 20. Find the sum of the first eight terms of the sequence.

(A) $\frac{178}{3}$ (B) $\frac{184}{3}$ (C) 38 *(D)* 40 (E) None of the answers (A)–(D) is correct.

Solution: Suppose we denote the two numbers x and y. Then

$$x + y = 2$$
$$x^2 + y^2 = 20.$$

Solving the first equation for x and substituting into the second equation results in $2y^2 - 4y - 16 = 0$, so $2(y - 4)(y + 2) = 0$, so 4 and -2. If $y = 4$, then $x = 2 - y = 2 - 4 = -2$. If $y = -2$, then $x = 2 - (-2) = 4$. Since the arithmetic sequence is increasing, we conclude that $a_1 = -2$ and $a_4 = 4$. Since

$$a_4 = a_1 + d + d + d = a_1 + 3d,$$

we know $4 = -2 + 3d$, so $3d = 6$, so $d = 2$. Therefore the sum of the first eight terms of this sequence is

$$-2 + 0 + 2 + 4 + 6 + 8 + 10 + 12 = 40.$$

27. One root of $f(x) = x^4 - 5x^3 + (6 - k)x^2 + 5kx - 6k$ is $x = 2$. There are two nonzero values of k, $(k_1 < k_2)$, for which the roots of $f(x)$ are not unique. What is their ratio, $\frac{k_2}{k_1}$?

(A) 9/4 (B) 3/2 (C) 2 (D) 4/3 (E) None of the answers (A)–(D) is correct.

Solution: Synthetic division shows that $f(x) = (x - 2)(x^3 - 3x^2 - kx + 3k)$. Factoring by grouping, we have

$$\begin{aligned} f(x) &= (x-2)(x^3 - 3x^2 - kx + 3k) \\ &= (x-2)\left(x^2(x-3) - k(x-3)\right) \\ &= (x-2)(x-3)(x^2 - k) \\ &= (x-2)(x-3)(x+\sqrt{k})(x-\sqrt{k}). \end{aligned}$$

Therefore, we need $k_1 = 4$ and $k_2 = 9$ in order to have a root with multiplicity. (The value $k = 0$ would also give a double root, but the problem specifies nonzero values of k.) Their ratio is 9/4.

28. Suppose that $f(x) = \dfrac{3x+1}{x-2}$ and $g(x) = x^3 + x - 2$. Note that both $f(x)$ and $g(x)$ are invertible. Calculate $\Big[(f^{-1}) \circ (g^{-1})\Big](-2)$.

(A) $\frac{4}{3}$ (B) $\frac{3}{5}$ *(C)* $-\frac{1}{3}$ (D) $-\frac{3}{2}$ (E) None of the answers (A)–(D) is correct.

Solution: First we compute $g^{-1}(-2)$. This equals the number x that satisfies $g(x) = -2$. By inspection, we see that $x = 0$.

Now we compute $f^{-1}(0)$. This equals the number x that satisfies $f(x) = 0$. Then $\frac{3x+1}{x-2} = 0$, so $3x + 1 = 0$, so $x = -\frac{1}{3}$.

Hence

$$\Big[(f^{-1}) \circ (g^{-1})\Big](-2) = f^{-1}\big(g^{-1}(-2)\big) = f^{-1}(0) = -\frac{1}{3}.$$

29. A twelve-hour digital watch displays the hours and minutes. For how long during one complete day does the watch display at least one 4?

(A) 1 hour 15 minutes (B) 5 hours and 24 minutes (C) 6 hours and 10 minutes

(D) 7 hours and 30 minutes (E) None of the answers (A)–(D) is correct.

Solution: In each hour, there are 15 minutes during which there is a 4 displayed in the minutes digits:

$$04, 14, 24, 34, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 54$$

So there are $24 \cdot 15 = 360$ minutes in the day during which the minutes digits contain a 4.

There are two hours in the day during which the hours digits contain a 4: the two 4 o'clock hours. This is $2 \cdot 60 = 120$ minutes.

During the two 4 o'clock hours, there are $2 \cdot 15 = 30$ minutes when both the hours digits and the minutes digits contain a 4.

Therefore, the total amount of time during which the watch displays at least one 4 is

$$360 + 120 - 30 = 450$$

minutes, which is 7 hours and 30 minutes.

30. Let $\lfloor x \rfloor$ denote the largest integer not exceeding x. Which of the following statements are always true?

$$\begin{aligned} I: &\quad \lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor \\ II: &\quad \lfloor 3x \rfloor = 3\lfloor x \rfloor \\ III: &\quad \lfloor -x \rfloor = -\lfloor x \rfloor \end{aligned}$$

(A) I only (B) II only (C) III only (D) all of the statements are true

(E) none of the statements are true

Solution:

- If $x = 1$ and $y = 1.1$ then $\lfloor x - y \rfloor = -1$ and $\lfloor x \rfloor - \lfloor y \rfloor = 0$.
- If $x = -1.1$ then $\lfloor 3x \rfloor = -4$ and $3\lfloor x \rfloor = -6$.
- If $x = 1.1$ then $\lfloor -x \rfloor = -2$ and $-\lfloor x \rfloor = -1$.

So none of the statements are true.