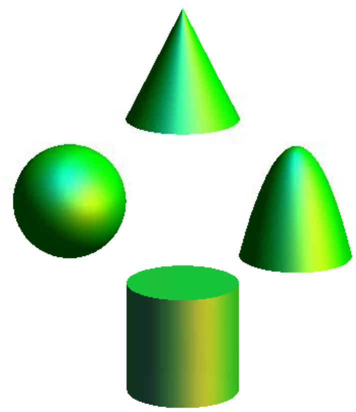
2017
High School Math Contest

Level 3 Exam Key



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1. The expression

$$\frac{\log_2 64}{\log_2 \sqrt{32}}$$

is equal to

(A) $\log_2(64 - \sqrt{32})$ (B) $\log_{32} 64$ (C) 1 *(D) 12/5*
(E) None of the answers (A) through (D) is correct.

Solution: We have

$$\frac{\log_2 64}{\log_2 \sqrt{32}} = \frac{\log_2 64}{(1/2)\log_2 32} = \frac{2(6)}{5} = \frac{12}{5}.$$

2. The number of five-digit numbers which remain unchanged when their digits are written in reverse order is

(A) 500 *(B) 900* (C) 1000 (D) 5000 (E) None of the answers (A) through (D) is correct.

Solution: The digits must be of the form $abcba$ where a is non-zero and b and c can be any digit. There are $9 \cdot 10 \cdot 10 = 900$ such combinations.

3. If the circle given by the equation $x^2 - 4x + y^2 - 8y - 44 = 0$ has a radius of r, find r.

(A) 8 (B) 16 (C) 32 (D) 64 (E) None of the choices (A) through (D) is correct.

Solution: Rewriting the circle in standard form gives $(x-2)^2 + (y-4)^2 = 64$, so $r = 8$.

4. Suppose you have a bag which contains discs marked 1, 2, 3, 4 and 5. You draw two discs from the bag, without replacement. What is the probability that their sum is odd?

(A) 1/2 *(B) 3/5* (C) 2/3 (D) 3/4 (E) None of the answers (A) through (D) is correct.

Solution: There are three odd numbers and two even numbers. To get an odd sum we need to draw an odd then an even, or an even then an odd. The probability of this occurring is

$$\frac{3 \cdot 2 + 2 \cdot 3}{5 \cdot 4} = \frac{3}{5}.$$

5. The sum of five numbers is 100. The sum of the first two numbers is 26, the sum of the second and third number is 48, the sum of the third and fourth number is 57, and the sum of the last two numbers is 32. The third number is

(A) 20 *(B) 42* (C) 15 (D) 9 (E) None of the answers (A) through (D) is correct.

Solution: We have

$$\begin{aligned} a+b+c+d+e &= 100, \\ a+b &= 26, \\ d+e &= 32. \end{aligned}$$

So

$$c = 100 - 26 - 32 = 42.$$

6. For which value of b does the function $f(x) = -x^2 + bx - 75$ have a maximum value of 25?

   (A) -50 (B) 50 *(C)* -20 (D) 30 (E) None of the answers (A) through (D) is correct.

Solution: The maximum value is at $x = \frac{-b}{2a} = b/2$. So the value of the maximum is

$$25 = f(b/2) = -\frac{b^2}{4} + \frac{b^2}{2} - 75.$$

So

$$100 = \frac{b^2}{4}$$

and

$$b = \pm\sqrt{400} = \pm 20.$$

So the answer is (C).

7. A cylinder has a height of 2 and a non-zero radius of r. If the cylinder has the same volume as a sphere with radius r, what is the value of r?

   (A) $\frac{\sqrt{2}}{8}$ (B) $\frac{3}{4\pi}$ *(C)* $\frac{3}{2}$ (D) $\frac{3}{8}$ (E) None of the choices (A) through (D) is correct.

Solution: Using the volume formulas we have

$$\pi r^2 h = \frac{4}{3}\pi r^3.$$

Assuming $h = 2$ and r is nonzero we have

$$2\pi = \frac{4}{3}\pi r$$

and $r = 3/2$.

8. Which of the following equations have *exactly* the same graph:

$$I : y = (x-1)(x+2)^2 \qquad II : y = x^3 + 3x - 4 \qquad III : y = \frac{(x-1)^2(x+2)^2(x-3)}{(x-1)(x-3)}$$

(A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) None of the answers (A) through (D) is correct.

Solution: The first and second graphs are different because

$$(x-1)(x+2)^2 = x^3+3x^2-4 \neq x^3+3x-4.$$

The third is different from both the first and the second because it has a different domain than either of the other two.

9. Let i be such that $i^2=-1$. Then

$$\frac{1}{1+\frac{1}{1+\frac{1}{i}}}$$

is equal to

(A) $\frac{3}{5}-\frac{i}{5}$ (B) $2-i$ (C) i (D) $\frac{1}{2}+\frac{2}{3}i$ (E) None of the answers (A) through (D) is correct.

Solution: We have

$$\begin{aligned}\frac{1}{1+\frac{1}{1+\frac{1}{i}}} &= \frac{1}{1+\frac{1}{1-i}}\\ &= \frac{1}{\frac{3}{2}+\frac{1}{2}i}\\ &= \frac{3}{5}-\frac{1}{5}i.\end{aligned}$$

10. Which of the following describes the solution x of the equation

$$3^{x+2} = \frac{81^{x-2}}{27(9^{x+1})}$$

(A) An odd integer (B) An even integer (C) A positive rational number which is not an integer
(D) A positive real number which is not a rational number
(E) None of the choices (A) through (D) is correct.

Solution: After simplifying we have

$$3^{x+2} = \frac{3^{4x-8}}{3^3 3^{2x+2}}.$$

Further simplification gives

$$x+2 = 4x-8-3-2x-2 = 2x-12.$$

The solution to this linear equation is is $x=15$.

11. If θ is an angle in the second quadrant with $\cos(\theta) = -\frac{2}{7}$, what is $\csc(\theta)$?

(A) $\dfrac{3\sqrt{5}}{7}$ (B) $-\dfrac{3\sqrt{5}}{7}$ *(C)* $\dfrac{7\sqrt{5}}{15}$ (D) $-\dfrac{7\sqrt{5}}{15}$
(E) None of the choices (A) through (D) is correct.

Solution: Since θ is in the second quadrant, $\csc(\theta)$ is positive. Using right angle trig we have

$$\sin(\theta) = \frac{\sqrt{45}}{7} = \frac{3\sqrt{5}}{7}.$$

Take the inverse and rationalize to obtain the correct answer.

12. Suppose b and c are real numbers. If $x^2 + bx + c$ has $2 + 3i$ as a root, what is $\dfrac{b}{c}$?

(A) $\dfrac{-2}{3}$ (B) $\dfrac{2}{3}$ *(C)* $\dfrac{-4}{13}$ (D) $\dfrac{4}{13}$ (E) None of the choices (A) through (D) is correct.

Solution: Since this is a monic polynomial with real coefficients, it can be written as

$$(x - (2 + 3i))(x - (2 - 3i)) = x^2 - 4x + 13.$$

13. Suppose $f(x) = 3x + 1$ and $g(x) = 5x - 2$. Which set gives the solutions of $|f(g(x)) - f(x)| \geq 2$.

(A) $\left(\dfrac{2}{3}, \infty\right)$ (B) $\left(\dfrac{1}{3}, \dfrac{2}{3}\right)$ (C) $\left(-\infty, \dfrac{-1}{3}\right) \cup \left(\dfrac{2}{3}, \infty\right)$ *(D)* $\left(-\infty, \dfrac{1}{3}\right] \cup \left[\dfrac{2}{3}, \infty\right)$
(E) None of the choices (A) through (D) is correct.

Solution: $|f(g(x)) - f(x)| = |12x - 6|$, so this is equivalent to finding the system of inequalities $12x - 6 \geq 2$ or $-12x + 6 \geq 2$. Solving the inequalities gives

$$x \geq 8/12 = 2/3$$
$$x \leq 4/12 = 1/3.$$

14. If $f(x) = x^4 + x - 1$ and n is a non-zero positive integer, which of the following must be true about $f(n)$?

(A) $f(n) \geq n$ (B) $f(n)$ is an odd integer (C) $f(n)$ is non-zero
(D) More than one of the choices (A) through (C) is correct
(E) None of the choices (A) through (D) is correct.

Solution: Choices (A) through (C) are all true. For (A) we have

$$n^4 + n - 1 \geq n$$
$$n^4 \geq 1.$$

Since n is a non-zero positive integer, this is always true.

For (B) if n is even then n^4 is also even so $n^4 + n - 1$ is an even plus an even minus 1, which is odd. If n is odd then n^4 is odd so $n^4 + n - 1$ is an odd plus an odd minus 1, which is odd.

For (C) if n is a non-zero positive integer then $n^4 \geq 0$ and $n \geq 1$ so

$$n^4 + n - 1 \geq n - 1 \geq 0.$$

Since all three options are true the answer is (D).

15. Suppose Cup A and Cup B both have water in them. A student walked into the room and decided to pour water between the two cups. First, she poured 2/3 of the water from Cup B into Cup A. She then poured 1/4 of the water from Cup A into Cup B. After the student performed these steps, Cup A had 21 liters of water and Cup B had 11 liters of water. How much water did Cup A contain before the student came into the room?

(A) 4 (B) 12 *(C) 20* (D) 28 (E) None of the choices (A) through (D) is correct.

Solution: Let a and b denote the original amounts. We know that $a + b = 32$. After the first stage we have $a + \frac{2}{3}b$ in Cup A. After the second stage we have

$$\frac{3}{4}\left(a + \frac{2}{3}b\right) = 21$$

in Cup A. Solving the system of equations gives $a = 20$ and $b = 12$.

16. One solution of the equation

$$0 = x^3 - 8x^2 + 16x - 3$$

is $x = 3$. Find the sum of the remaining solutions.

(A) 5 (B) $\frac{5}{2}$ (C) $\frac{5 + 2\sqrt{21}}{2}$ (D) 8 (E) None of the answers (A) through (D) is correct.

Solution: Since $x = 3$ is a solution, $(x - 3)$ must be a factor. Using polynomial division we have

$$0 = (x - 3)(x^2 - 5x + 1).$$

So the remaining solutions are given by the quadratic formula. We have

$$x = \frac{5 \pm \sqrt{5^2 - 4}}{2}.$$

Of course, after adding the two solutions together we get 5.

17. Solve the following equation:

$$9^x - 3^{x+1} = 10$$

(A) $x = 5$ (B) $x = -2$ (C) $x = \log_3 -2$ *(D) $x = \log_3 5$*
(E) None of the answers (A) through (D) is correct.

Solution: We have

$$(3^x)^2 - 3(3^x) - 10 = 0$$

so that

$$(3^x - 5)(3^x + 2) = 0.$$

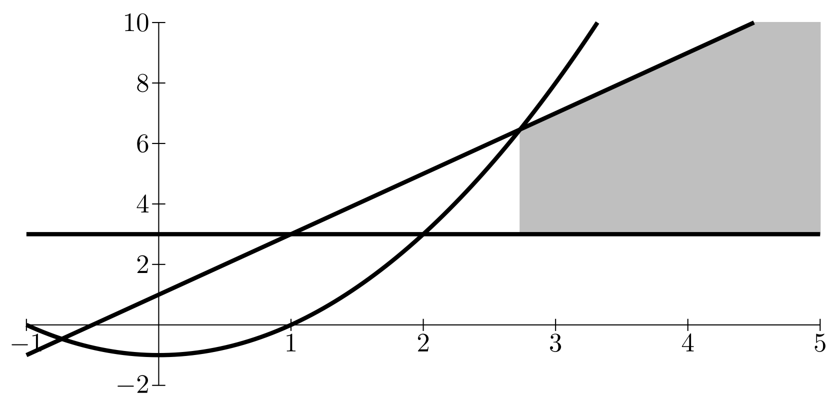
The only possibility is $3^x - 5 = 0$ or $x = \log_3 5$.

18. Suppose $f(x) = x^a$ where a is a positive real number. If $f(f(f(2))) = 64$ then a satisfies

(A) $0 < a < 1$ (B) $a = 1$ *(C) $1 < a < 2$* (D) $a = 2$ (E) $a > 2$

Solution: If $a = 1$ then $f(f(f(2))) = 2 < 64$. If $a = 2$ then $f(f(f(2))) = 256 > 64$. Since 2^a is monotonically increasing with respect to a this implies $1 < a < 2$.

19. Which of the following inequalities has its solution represented by the shaded region in the graph below?



(A) $3 \le x^2 - 1 \le y \le 2x + 1$ (B) $x^2 - 1 \le 2x + 1 \le y \le 3$ (C) $3 \le 2x + 1 \le y \le x^2 - 1$
(D) $3 \le y \le 2x + 1 \le x^2 - 1$ (E) None of the answers (A) through (D) is correct.

Solution: The region is exactly where the line $2x + 1$ is bigger than 3 but less than the parabola and the shaded area is between $y = 3$ and $y = 2x + 1$.

20. The sum of all the digits appearing in the first fifty positive integers is

(A) 225 *(B) 330* (C) 1275 (D) 150 (E) None of the answers (A) through (D) is correct.

Solution: The ones digit has the numbers 0 to 9 repeated 5 times. The total is

$$5 \cdot (0 + 1 + 2 + \ldots + 9) = 225.$$

For the tens digit we have ten 1's, ten 2's, ten 3's, ten 4's and one 5. So the cumulative total is

$$225 + 10(1 + 2 + 3 + 4) + 5 = 330.$$

21. One store sells AA batteries for \$2 per battery and AAA batteries for \$1 per battery. A second store sells AA batteries for \$3 per battery and AAA batteries for \$4 per battery. You buy n AA batteries and m AAA batteries from each store and you spend \$30 overall. How many batteries total did you buy?

(A) 6 (B) 10 *(C) 12* (D) 24 (E) None of the answers (A) through (D) is correct.

Solution: We have

$$\begin{aligned} 2n + m + 3n + 4m &= 30 \\ 5(n + m) &= 30. \end{aligned}$$

So $n + m = 6$. This means we bought 6 batteries from each store, and 12 overall.

22. The roots of the equation $(\sin\theta + \cos\theta)^2 = 0$ are

(A) all multiples of π (B) all odd multiples of π (C) all odd multiples of $\pi/2$
(D) all odd multiples of $\pi/4$ *(E) None of the answers (A) through (D) is correct.*

Solution: We have

$$\begin{aligned} 0 &= (\cos\theta + \sin\theta)^2 \\ &= \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta \\ &= 1 + \sin(2\theta). \end{aligned}$$

So

$$\sin(2\theta) = -1,$$

and $\theta = 3\pi/4 + n\pi$ where n is any integer. This does not match any of the solutions.

23. Each of ten users in an internet chat room claims that exactly two of the other nine are 30 years old or older. Not all of them are lying. The number of them telling the truth is

(A) 3 (B) 5 (C) 6 (D) 8 *(E) Not enough information to determine.*

Solution: At least one person is telling the truth. If that person is an under 30 then there are 8 people younger than 30 and 2 people over in the chat room. The 8 people under 30 are telling the truth and the two people over 30 are lying. If that person is over 30 then there are 3 people over 30 and 7 people under 30 in the chat room. The 3 people over 30 are telling the truth and the 7 people under 30 are lying. Since we have two consistent answers we can't determine the number of people telling the truth.

24. For what values of a and b is the equation

$$\left(a^{\ln b}\right)^{ab} = \left(b^{\ln a}\right)^{ba},$$

where ln denotes the natural log, well defined and true?

(A) no values *(B) all non-zero positive values* (C) all non-negative values (D) all values
(E) None of the answers (A) through (D) is correct.

Solution: Note that the equation doesn't make sense for non-zero positive values of a and b because of the domain of natural log. If a and b are positive then we have

$$\begin{aligned} a^{ab\ln(b)} &= b^{ab\ln(a)} \\ \ln(a^{ab\ln(b)}) &= \ln(b^{ab\ln(a)}) \\ ab\ln(a)\ln(b) &= ab\ln(a)\ln(b) \end{aligned}$$

So the equation holds for all non-zero positive values.

25. A linear function of the form $f(x) = 3x + b$ has an inverse function of the form $f^{-1}(x) = ax + k$. Which of the following is equal to the product ak?

(A) $\frac{-b}{3}$ (B) $\frac{b}{3}$ *(C)* $\frac{-b}{9}$ (D) $\frac{b}{9}$ (E) None of the choices (A) through (D) is correct.

Solution: Solving for the inverse we have

$$x = \frac{y-b}{3} = \frac{1}{3}y - \frac{b}{3}.$$

So $a = \frac{1}{3}$ and $k = -\frac{b}{3}$.

26. A certain rectangle has vertices $(1, 3)$, $(2, 4)$, $(6, 0)$ and (a, b). What is the value of $a + b$?

(A) 0 *(B) 4* (C) 5 (D) 6 (E) None of the choices (A) through (D) is correct.

Solution: Since the sides must be parallel and have the same length, the change from $(1, 3)$ to $(2, 4)$ must be the same as the change from (a, b) to $(6, 0)$. So $(a, b) = (5, -1)$.

27. We do not know the formula for $f(x)$, but we know that it satisfies $f(2x + 3) = f(x) + 5$ for any real number x. What is the slope of the line that intersects the graph of $y = f(x)$ at $x = 1$ and at $x = 13$?

(A) $\frac{5}{2}$ (B) $\frac{5}{3}$ *(C)* $\frac{5}{6}$ (D) $\frac{5}{13}$ (E) None of the choices (A) through (D) is correct.

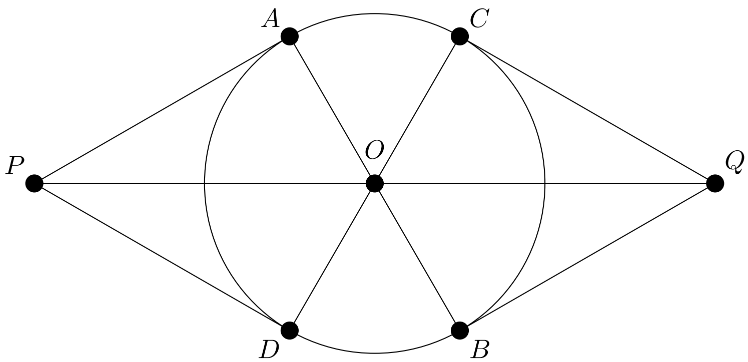
Solution: Since $f(2x + 3) = f(x) + 5$ we have

$$\begin{aligned} f(5) &= f(1) + 5 \\ f(13) &= f(5) + 5. \end{aligned}$$

Thus $f(13) = f(1) + 10$ and the slope of the secant line is

$$m = \frac{f(13) - f(1)}{13 - 1} = \frac{10}{12} = \frac{5}{6}.$$

28. Consider the figure shown below.



In the figure above, the center O of the circle is on all three line segments $\overline{PQ}$, $\overline{AB}$, and $\overline{CD}$. Also the line segments $\overline{AP}$, $\overline{DP}$, $\overline{CQ}$, and $\overline{BQ}$ are all tangents to the circle.

If $AB = 2$ and $\angle AOC = 60°$ then PQ is equal to

(A) 4 (B) 3 (C) 2 (D) 1 (E) None of the answers (A) through (D) is correct.

Solution: First note that triangles formed by lines tangent to a circle are right triangles with the right angle at the point of tangency. Since the right triangles $\triangle OCQ$ and $\triangle OBD$ have two equal sides they are congruent. This implies that $\angle COQ = \angle QOB$. Since

$$\angle AOC + \angle COQ + \angle QOB = 180°$$

we must have $2\angle COQ = 120°$ and $\angle COQ = 60°$. Note that the diameter of the circle has length 2 so the radius must be length 1. Using the fact that

$$\cos(\angle COQ) = \frac{OC}{OQ}$$

we have

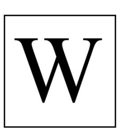
$$OQ = \frac{OC}{\cos(\angle COQ)} = \frac{1}{\cos(60°)} = 2.$$

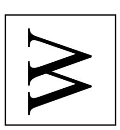
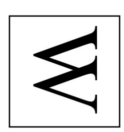
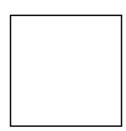
It follows from symmetry that $PO = 2$ and $PQ = 4$.

29. A certain university has a logo that is a rotating cube in which one face has a W on it and the other five faces are blank. Originally the W-face is at the front of the cube as shown in (A) below. Then the following sequence of moves are repeated over and over:

1. rotate the cube 90° around a horizontal axis, so that the front face moves counter clockwise;
2. rotate the cube 90° around a horizontal axis, so that the front face moves down;
3. rotate the cube 90° around the vertical axis, so that the front face moves to the left;
4. rotate the cube 90° around the same horizontal axis, and in the same direction, as step 2.

After this sequence has been repeated a total of 2017 times the front face will look like

(A) 

(B) 
(C) 
(D) 
(E) 

Solution: After you perform the sequence of moves once you end up with the logo on the back face of the cube, and after you perform them again you get the logo on the front face of the cube in the original orientation. However, since we are repeating the steps an odd number of times, we will end up with a blank face on the front.

30. When $x^{57} - 3x^{21} + 6$ is divided by $2x + 2$ the remainder is

(A) 2 (B) 4 (C) 6 *(D) 8* (E) None of the answers (A) through (D) is correct.

Solution: Let $u = 2x$ so that $x = \frac{u}{2}$. Then we are dividing

$$\left(\frac{u}{2}\right)^{57} - 3\left(\frac{u}{2}\right)^{21} + 6$$

by $u + 2$. The Remainder Theorem says that this is equal to

$$\left(\frac{-2}{2}\right)^{57} - 3\left(\frac{-2}{2}\right)^{21} + 6 = -1 + 3 + 6 = 8.$$