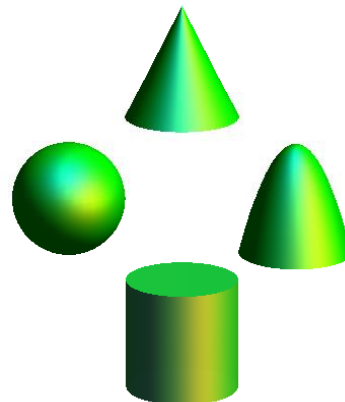




2017
High School Math Contest

Level 3
Exam



Lenoir-Rhyne University

*Donald and Helen Schort School of
Mathematics and Computing Sciences*

This exam has been prepared by the following faculty from **Western Carolina University**:

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Do's and Don'ts:

- **Do NOT** open this booklet until you are instructed to do so.
- **NO Calculators** (Or other electronic devices)
- Contestants with electronic devices (on or off) will be **disqualified!**
And their **team** will be **disqualified!**
(The other team members will continue to participate in the individual contest.)

DIRECTIONS:

- **The Score Card:**
Write:
 - Your **NAME** on the “name line” (of course).
 - “**Level 3**” on the “subject line”.
 - Your **SCHOOL** on the “**Date Line**”
 - **Clearly mark** ONE bubble using **#2 PENCIL**.
 - Light marks will be counted as unmarked!
 - **Completely erase** any changes.
 - You **can** write on this test booklet. (But the test booklet will not be graded.)
 - **Tie Breakers:** In case of ties, the person with the least number of *wrong answers* wins.
(A *blank* is better than *incorrect!*)
 - **The Exam:** **30** problems, **70** minutes.
- WAIT** for the signal to begin.

1. The expression

$$\frac{\log_2 64}{\log_2 \sqrt{32}}$$

is equal to

- (A) $\log_2(64 - \sqrt{32})$ (B) $\log_{32} 64$ (C) 1 (D) $12/5$
(E) None of the answers (A) through (D) is correct.

2. The number of five-digit numbers which remain unchanged when their digits are written in reverse order is

- (A) 500 (B) 900 (C) 1000 (D) 5000 (E) None of the answers (A) through (D) is correct.

3. If the circle given by the equation $x^2 - 4x + y^2 - 8y - 44 = 0$ has a radius of r , find r .

- (A) 8 (B) 16 (C) 32 (D) 64 (E) None of the choices (A) through (D) is correct.

4. Suppose you have a bag which contains discs marked 1, 2, 3, 4 and 5. You draw two discs from the bag, without replacement. What is the probability that their sum is odd?

- (A) $1/2$ (B) $3/5$ (C) $2/3$ (D) $3/4$ (E) None of the answers (A) through (D) is correct.

5. The sum of five numbers is 100. The sum of the first two numbers is 26, the sum of the second and third number is 48, the sum of the third and fourth number is 57, and the sum of the last two numbers is 32. The third number is

- (A) 20 (B) 42 (C) 15 (D) 9 (E) None of the answers (A) through (D) is correct.

6. For which value of b does the function $f(x) = -x^2 + bx - 75$ have a maximum value of 25?

- (A) -50 (B) 50 (C) -20 (D) 30 (E) None of the answers (A) through (D) is correct.

7. A cylinder has a height of 2 and a non-zero radius of r . If the cylinder has the same volume as a sphere with radius r , what is the value of r ?

- (A) $\frac{\sqrt{2}}{8}$ (B) $\frac{3}{4\pi}$ (C) $\frac{3}{2}$ (D) $\frac{3}{8}$ (E) None of the choices (A) through (D) is correct.

8. Which of the following equations have *exactly* the same graph:

$$I : y = (x - 1)(x + 2)^2 \quad II : y = x^3 + 3x - 4 \quad III : y = \frac{(x - 1)^2(x + 2)^2(x - 3)}{(x - 1)(x - 3)}$$

- (A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) None of the answers (A) through (D) is correct.

9. Let i be such that $i^2 = -1$. Then

$$\frac{1}{1 + \frac{1}{1+i}}$$

is equal to

- (A) $\frac{3}{5} - \frac{i}{5}$ (B) $2 - i$ (C) i (D) $\frac{1}{2} + \frac{2}{3}i$ (E) None of the answers (A) through (D) is correct.

10. Which of the following describes the solution x of the equation

$$3^{x+2} = \frac{81^{x-2}}{27(9^{x+1})}$$

- (A) An odd integer (B) An even integer (C) A positive rational number which is not an integer
(D) A positive real number which is not a rational number
(E) None of the choices (A) through (D) is correct.

11. If θ is an angle in the second quadrant with $\cos(\theta) = -\frac{2}{7}$, what is $\csc(\theta)$?

- (A) $\frac{3\sqrt{5}}{7}$ (B) $-\frac{3\sqrt{5}}{7}$ (C) $\frac{7\sqrt{5}}{15}$ (D) $-\frac{7\sqrt{5}}{15}$
(E) None of the choices (A) through (D) is correct.

12. Suppose b and c are real numbers. If $x^2 + bx + c$ has $2 + 3i$ as a root, what is $\frac{b}{c}$?

- (A) $-\frac{2}{3}$ (B) $\frac{2}{3}$ (C) $-\frac{4}{13}$ (D) $\frac{4}{13}$ (E) None of the choices (A) through (D) is correct.

13. Suppose $f(x) = 3x + 1$ and $g(x) = 5x - 2$. Which set gives the solutions of $|f(g(x)) - f(x)| \geq 2$.

- (A) $\left(\frac{2}{3}, \infty\right)$ (B) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (C) $\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$ (D) $\left(-\infty, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \infty\right)$
(E) None of the choices (A) through (D) is correct.

14. If $f(x) = x^4 + x - 1$ and n is a non-zero positive integer, which of the following must be true about $f(n)$?

- (A) $f(n) \geq n$ (B) $f(n)$ is an odd integer (C) $f(n)$ is non-zero
(D) More than one of the choices (A) through (C) is correct
(E) None of the choices (A) through (D) is correct.

15. Suppose Cup A and Cup B both have water in them. A student walked into the room and decided to pour water between the two cups. First, she poured $\frac{2}{3}$ of the water from Cup B into Cup A. She then poured $\frac{1}{4}$ of the water from Cup A into Cup B. After the student performed these steps, Cup A had 21 liters of water and Cup B had 11 liters of water. How much water did Cup A contain before the student came into the room?

- (A) 4 (B) 12 (C) 20 (D) 28 (E) None of the choices (A) through (D) is correct.

16. One solution of the equation

$$0 = x^3 - 8x^2 + 16x - 3$$

is $x = 3$. Find the sum of the remaining solutions.

- (A) 5 (B) $\frac{5}{2}$ (C) $\frac{5 + 2\sqrt{21}}{2}$ (D) 8 (E) None of the answers (A) through (D) is correct.

17. Solve the following equation:

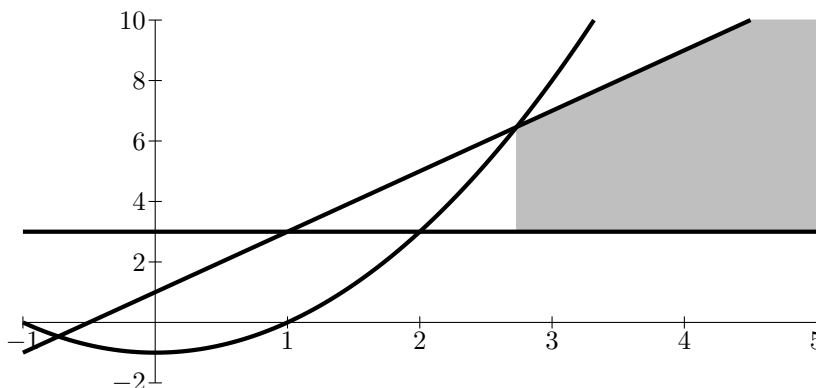
$$9^x - 3^{x+1} = 10$$

- (A) $x = 5$ (B) $x = -2$ (C) $x = \log_3 -2$ (D) $x = \log_3 5$
(E) None of the answers (A) through (D) is correct.

18. Suppose $f(x) = x^a$ where a is a positive real number. If $f(f(f(2))) = 64$ then a satisfies

- (A) $0 < a < 1$ (B) $a = 1$ (C) $1 < a < 2$ (D) $a = 2$ (E) $a > 2$

19. Which of the following inequalities has its solution represented by the shaded region in the graph below?



- (A) $3 \leq x^2 - 1 \leq y \leq 2x + 1$ (B) $x^2 - 1 \leq 2x + 1 \leq y \leq 3$ (C) $3 \leq 2x + 1 \leq y \leq x^2 - 1$
(D) $3 \leq y \leq 2x + 1 \leq x^2 - 1$ (E) None of the answers (A) through (D) is correct.

20. The sum of all the digits appearing in the first fifty positive integers is

- (A) 225 (B) 330 (C) 1275 (D) 150 (E) None of the answers (A) through (D) is correct.

21. One store sells AA batteries for \$2 per battery and AAA batteries for \$1 per battery. A second store sells AA batteries for \$3 per battery and AAA batteries for \$4 per battery. You buy n AA batteries and m AAA batteries from each store and you spend \$30 overall. How many batteries total did you buy?

- (A) 6 (B) 10 (C) 12 (D) 24 (E) None of the answers (A) through (D) is correct.

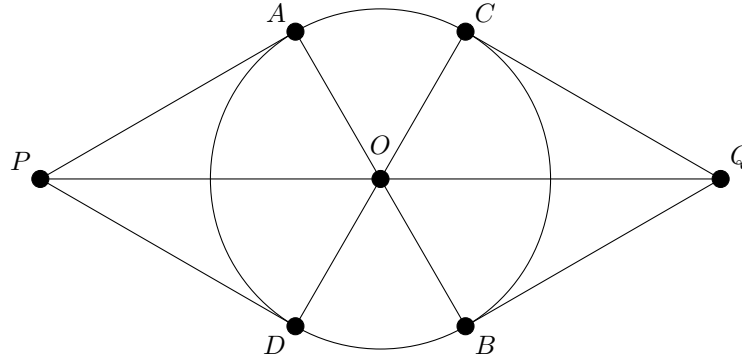
22. The roots of the equation $(\sin \theta + \cos \theta)^2 = 0$ are
 (A) all multiples of π (B) all odd multiples of π (C) all odd multiples of $\pi/2$
 (D) all odd multiples of $\pi/4$ (E) None of the answers (A) through (D) is correct.
23. Each of ten users in an internet chat room claims that exactly two of the other nine are 30 years old or older. Not all of them are lying. The number of them telling the truth is
 (A) 3 (B) 5 (C) 6 (D) 8 (E) Not enough information to determine.
24. For what values of a and b is the equation

$$(a^{\ln b})^{ab} = (b^{\ln a})^{ba},$$

where \ln denotes the natural log, well defined and true?

- (A) no values (B) all non-zero positive values (C) all non-negative values (D) all values
 (E) None of the answers (A) through (D) is correct.
25. A linear function of the form $f(x) = 3x + b$ has an inverse function of the form $f^{-1}(x) = ax + k$. Which of the following is equal to the product ak ?
 (A) $\frac{-b}{3}$ (B) $\frac{b}{3}$ (C) $\frac{-b}{9}$ (D) $\frac{b}{9}$ (E) None of the choices (A) through (D) is correct.
26. A certain rectangle has vertices $(1, 3)$, $(2, 4)$, $(6, 0)$ and (a, b) . What is the value of $a + b$?
 (A) 0 (B) 4 (C) 5 (D) 6 (E) None of the choices (A) through (D) is correct.
27. We do not know the formula for $f(x)$, but we know that it satisfies $f(2x + 3) = f(x) + 5$ for any real number x . What is the slope of the line that intersects the graph of $y = f(x)$ at $x = 1$ and at $x = 13$?
 (A) $\frac{5}{2}$ (B) $\frac{5}{3}$ (C) $\frac{5}{6}$ (D) $\frac{5}{13}$ (E) None of the choices (A) through (D) is correct.

28. Consider the figure shown below.



In the figure above, the center O of the circle is on all three line segments \overline{PQ} , \overline{AB} , and \overline{CD} . Also the line segments \overline{AP} , \overline{DP} , \overline{CQ} , and \overline{BQ} are all tangents to the circle.





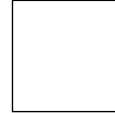
If $AB = 2$ and $\angle AOC = 60^\circ$ then PQ is equal to

- (A) 4 (B) 3 (C) 2 (D) 1 (E) None of the answers (A) through (D) is correct.

29. A certain university has a logo that is a rotating cube in which one face has a W on it and the other five faces are blank. Originally the W-face is at the front of the cube as shown in (A) below. Then the following sequence of moves are repeated over and over:

1. rotate the cube 90° around a horizontal axis, so that the front face moves counter clockwise;
2. rotate the cube 90° around a horizontal axis, so that the front face moves down;
3. rotate the cube 90° around the vertical axis, so that the front face moves to the left;
4. rotate the cube 90° around the same horizontal axis, and in the same direction, as step 2.

After this sequence has been repeated a total of 2017 times the front face will look like

- (A)  (B)  (C)  (D)  (E) 

30. When $x^{57} - 3x^{21} + 6$ is divided by $2x + 2$ the remainder is

- (A) 2 (B) 4 (C) 6 (D) 8 (E) None of the answers (A) through (D) is correct.