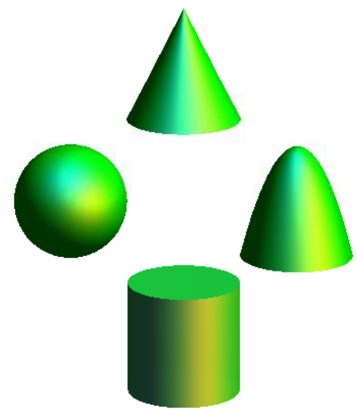
2015
High School Math Contest

Level 2 Exam Key



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1. A circle is unwound and re-shaped into a square. Find the ratio of the square's area to the circle's area.

   (A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{\sqrt{2}}$ **(E)** $\frac{4}{\pi}$

   Solution: Let r be the radius of the circle. Then the circle's area is πr^2 and its circumference is $2\pi r$. The circumference of the circle is the same as the perimeter of the square, so the square has side length $\frac{2\pi r}{4} = \frac{\pi r}{2}$. Therefore the area of the square is $\left(\frac{\pi r}{2}\right)^2 = \frac{\pi^2 r^2}{4}$. Hence the ratio of the square's area to the circle's area is

   $$\frac{\pi r^2}{\pi^2 r^2/4} = \frac{4}{\pi}.$$

2. Given $f(x) = x + 2$ and $g(x) = \sqrt[3]{x}$, find $f^{-1}\big(g^{-1}(2)\big)$.

   (A) 8 **(B)** 6 (C) 2 (D) -2 (E) -6

   Solution: We have $f^{-1}(x) = x - 2$ and $g^{-1}(x) = x^3$, so

   $$(f^{-1} \circ g^{-1})(2) = f^{-1}\big(g^{-1}(2)\big) = f^{-1}(2^3) = f^{-1}(8) = 8 - 2 = 6.$$

3. An isosceles triangle $\triangle ABC$ has a smallest angle A that is one-third of the size of the smallest of the remaining angles — (that is, the smallest of B and C). What is the approximate size of angle A?

   (A) 23.0° **(B)** 25.7° (C) 45° (D) 77.1° (E) None of the answers (A)–(D) is correct.

   Solution: Let a, b, c denote the measures of angles A, B, C, respectively, in degrees. Then $a+b+c = 180$, and because the triangle is isosceles and a is one-third the smallest of b and c, then $b = c$ and we have that $180 = a + b + c = \frac{b}{3} + 2b$. Solving for b yields $\frac{7b}{3} = 180$ and so $b = \frac{540}{7}$. Substituting back in for a, we get $a + \frac{1080}{7} = 180$, so $a \approx 25.7$.

4. One solution of $x^3 + 5x^2 - 2x - 4 = 0$ is $x = 1$. Which of the following is another solution?

   (A) $-1 + \sqrt{7}$ **(B)** $-3 + \sqrt{5}$ (C) $-2 + \sqrt{5}$ (D) $-3 + \sqrt{3}$ (E) $-5 + \sqrt{2}$

   Solution: By long division, we see that

   $$x^3 + 5x^2 - 2x - 4 = (x - 1)(x^2 + 6x + 4).$$

   The quadratic formula tells us that the roots of $x^2 + 6x + 4$ are $x = -3 + \sqrt{5}$ and $x = -3 = \sqrt{5}$.

5. If a, b, and c are real numbers such that $\frac{a}{b} = 3$ and $\frac{b}{c} = 7$, then $\frac{a+b}{b+c}$ equals:

(A) $\frac{7}{2}$ (B) $\frac{7}{8}$ (C) $\frac{3}{7}$ (D) $\frac{1}{7}$ (E) 21

> **Solution:** From the given information, we know that $a = 3b$ and $b = 7c$. Hence $a = 3b = 3(7c) = 21c$, so
> $$\frac{a+b}{b+c} = \frac{21c+7c}{7c+c} = \frac{28c}{8c} = \frac{7}{2}.$$

6. Find the sum of all values of m that make the polynomial $x^2 + (m+5)x + (5m+1)$ a perfect square.

(A) 3 (B) 4 (C) 7 (D) 8 **(E)** 10

> **Solution:** In order for the polynomial to be a perfect square, the constant coefficient must equal the square of half the coefficient of x:
> $$\left(\frac{m+5}{2}\right)^2 = 5m+1$$
> $$\frac{m^2+10m+25}{4} = 5m+1$$
> $$m^2+10m+25 = 20m+4$$
> $$m^2-10m+21 = 0$$
> $$(m-3)(m-7) = 0$$
> Hence $m = 3$ or $m = 7$, and the sum of these values is $3+7 = 10$.

7. Find the sum of all real values of x that satisfy the equation $\big||x-2|-3\big| = 4$.

(A) -8 (B) -4 **(C)** 4 (D) 8 (E) None of the answers (A)–(D) is correct.

> **Solution:** We know that $|x-2|-3 = 4$ or $|x-2|-3 = -4$. Hence $|x-2| = 7$ or $|x-2| = -1$. Since $|x-2|$ cannot be negative, this means $|x-2| = 7$. Therefore $x-2 = 7$ or $x-2 = -7$, so $x = 9$ or $x = -5$. The sum of these two solutions is $9 + (-5) = 4$.

8. Suppose a and b are the solutions to $9y^2 + 9y = 4$. What is $2|a-b| + |ab|$?

(A) $\frac{14}{3}$ (B) $-\frac{34}{9}$ (C) $-\frac{14}{3}$ **(D)** $\frac{34}{9}$ (E) None of the answers (A)–(D) is correct.

Solution: We have:

$$9y^2 + 9y = 4$$
$$9y^2 + 9y - 4 = 0$$
$$(3y - 1)(3y + 4) = 0$$

Hence $3y - 1 = 0$ or $3y + 4 = 0$, so $y = \frac{1}{3}$ or $y = -\frac{4}{3}$.
Let $a = \frac{1}{3}$ and $b = -\frac{4}{3}$. Then

$$\begin{aligned} 2|a-b| + |ab| &= 2\left|\frac{1}{3} - \left(-\frac{4}{3}\right)\right| + \left|\frac{1}{3} \cdot -\frac{4}{3}\right| \\ &= 2\left|\frac{1}{3} + \frac{4}{3}\right| + \left|-\frac{4}{9}\right| \\ &= 2\left|\frac{5}{3}\right| + \frac{4}{9} \\ &= \frac{10}{3} + \frac{4}{9} \\ &= \frac{30}{9} + \frac{4}{9} \\ &= \frac{34}{9}. \end{aligned}$$

9. To get from point A to point B you must avoid walking through a swamp. To avoid the swamp, you must walk 34 meters south and 41 meters east. To the nearest meter, how many meters would be saved if it were possible to walk through the swamp?

(A) 22 (B) 34 (C) 53 (D) 75 (E) None of the answers (A)–(D) is correct.

Solution: By the Pythagorean theorem, we have

$$\bar{AB}^2 = 34^2 + 41^2 = 1156 + 1681 = 2837.$$

So $\sqrt{2837} = \bar{AB}$. At this point, a little guess and test should show that $\sqrt{2837} \approx 53$. To go around the swamp, you must walk $34 + 41 = 75$ meters. Hence, if it were possible to walk directly through the swamp, the number of meters saved would be approximately $75 - 53 = 22$

10. Suppose $h(x) = x^2 - 2x$. What is an equation of the secant line containing $\big(2, h(2)\big)$ and $\big(4, h(4)\big)$?

(A) $y = 4x - 8$ (B) $y = 10x - 16$ (C) $y = 4x$ (D) $y = 16x$

(E) None of the answers (A)–(D) is correct.

Solution: We compute $h(2) = 2^2 - 2(2) = 0$ and $h(4) = 4^2 - 2(4) = 8$. Hence the slope of the secant line is

$$m = \frac{h(4) - h(2)}{4 - 2} = \frac{8 - 0}{2} = 4.$$

Since the line contains the point $\big(4, h(4)\big) = (4, 0)$, it has equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= 4(x - 2) \\ y &= 4x - 8 \end{aligned}$$

11. Which of the following polynomials has roots 4, -3, and -1?

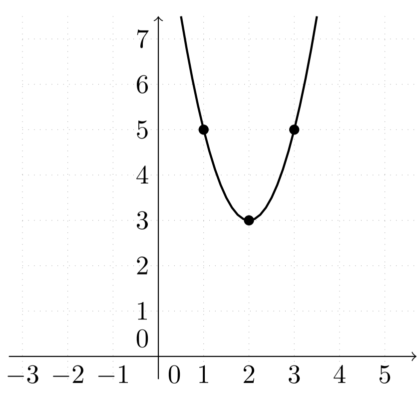
(A) $y = x^3 - 8x^2 + 19x - 12$ **(B)** $y = x^3 - 13x - 12$ (C) $y = x^3 + 8x^2 + 19x + 12$ (D) $y = x^3 - 13x + 12$

(E) None of the answers (A)–(D) is correct.

Solution: One polynomial with these roots is

$$\begin{aligned} y &= (x - 4)(x + 3)(x + 1) \\ &= (x^2 - x - 12)(x + 1) \\ &= x^3 - x^2 - 12x + x^2 - x - 12 \\ &= x^3 - 13x - 12. \end{aligned}$$

12. Suppose $f(x) = x^2$. Below is a graph of $g(x)$, which involves transformations of $f(x)$. Using the graph below, what is $g(x)$ in terms of transforming $f(x)$?



(A) $2f(x - 2) + 3$ (B) $\frac{1}{2}f(x + 2) + 3$ (C) $2f(x + 2) + 3$ (D) $\frac{1}{2}f(x - 2) + 3$

(E) None of the answers (A)–(D) is correct.

Solution: The graph of $g(x)$ is the same as the graph of $f(x)$ shifted up 3 units, right 2 units and stretched by a factor of 2. Therefore, $g(x) = 2f(x-2)+3$.

13. What is the sum of the solutions of $\sqrt{3x+1} - \sqrt{x-1} = 2$?

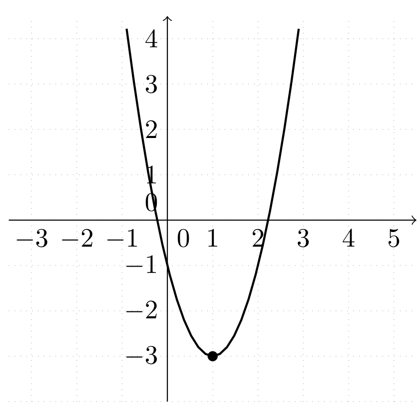
(A) 1 (B) 5 **(C) 6** (D) 7 (E) None of the answers (A)–(D) is correct.

Solution:

$$\begin{aligned}
\sqrt{3x+1} - \sqrt{x-1} &= 2 \\
\sqrt{3x+1} &= 2 + \sqrt{x-1} \\
(\sqrt{3x+1})^2 &= (2+\sqrt{x-1})^2 \\
3x+1 &= 4 + 4\sqrt{x-1} + x - 1 \\
2x-2 &= 4\sqrt{x-1} \\
(2x-2)^2 &= (4\sqrt{x-1})^2 \\
4x^2 - 8x + 4 &= 16(x-1) \\
4x^2 - 8x + 4 &= 16x - 16 \\
4x^2 - 24x + 20 &= 0 \\
x^2 - 6x + 5 &= 0 \\
(x-5)(x-1) &= 0
\end{aligned}$$

Therefore $x-5=0$ or $x-1=0$, so $x=5$ or $x=1$. Quick substitutions into the original equation reveal that these really are both solutions. Hence the sum of the solutions is $5+1=6$.

14. Consider the two quadratic functions $f(x)$ and $g(x)$, where $f(x)$ is shown below and $g(x) = x^2 + 2x - 8$. What is the y-coordinate of the lower minimum of the two functions?



(A) 1 (B) −1 (C) −3 (D) −12 **(E) None of the answers (A)–(D) is correct.**

Solution: The vertex of $g(x)$ has x-coordinate

$$x = -\frac{b}{2a} = -\frac{2}{2(1)} = -1,$$

so its y-coordinate is

$$g(-1) = (-1)^2 + 2(-1) - 8 = -9.$$

So the vertex for $g(x)$ is at (-1,-9), while the vertex for the graph is at $(1, -3)$. The y-coordinate of the vertex with the lower minimum is -9. Since -9 is not one of the options, the answer is "None of the answers (A)–(D) is correct."

15. Real estate ads suggest that 53% of homes for sale have garages, 25% have swimming pools, and 4% have both features. Let a denote the percentage that have a garage or swimming pool or both. Let b denote the percentage that have a garage but not a pool. Let c denote the percentage that have a pool but not a garage. Find $b + c - a$.

(A) -4% (B) 0% (C) 4% (D) 8% (E) None of the answers (A)–(D) is correct.

Solution: Observe that the collection of people who have a garage or a pool (inclusively), consists of those who have a garage but not a pool, those who have a pool but not a garage, and those who have both a garage and a pool. Furthermore, these three groups of people do not overlap. Therefore $a = b + c + 4\%$. Rearranging this, we have

$$b + c - a = -4\%.$$

16. Given that $5^{x+1} = 30$, what is the value of 5^{3x+1}?

(A) $\log_5 30 - 1$ (B) 90 (C) 180 **(D) 1080** (E) None of the answers (A)–(D) is correct.

Solution: Since $30 = 5^{x+1} = (5^x)(5)$, it follows that $5^x = 30/5 = 6$. We then write $5^{3x+1} = (5^x)(5^x)(5^x)(5^1) = (6)(6)(6)(5) = 1080$.

17. Three concentric circles have radii 2, 3, and 4 inches, respectively. What percent of the area of the largest circle is the area of the middle ring, rounded to the nearest whole percent?

(A) 12.5% (B) 25% **(C)** 31% (D) 33% (E) None of the answers (A)–(D) is correct.

Solution: The area of the middle ring is the area of the middle circle minus the area of the smallest circle, which is

$$\pi(3)^2 - \pi(2)^2 = 9\pi - 4\pi = 5\pi.$$

The area of the largest circle is $\pi(5)^2 = 25\pi$. Therefore the proportion of the area of the middle ring to the area of the largest circle is

$$\frac{5\pi}{16\pi} = \frac{5}{16} \approx 0.3125 \approx 31\%.$$

18. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

(A) 11 (B) 12 (C) 13 (D) 14 **(E) None of the answers (A)–(D) is correct.**

Solution: Let d be the length of one lap in miles. Then he needs to complete the four laps in $\frac{4d}{10} = \frac{2d}{5}$ hours. He has already spent $\frac{3d}{9} = \frac{d}{3}$ hours on the first three laps, so he has $\frac{2d}{5} - \frac{d}{3} = \frac{d}{15}$ hours left. Therefore, he must run with a speed of

$$\frac{d}{d/15} = 15$$

mph on the final lap. Hence, the answer is "None of the answers (A)–(D) is correct."

19. A basketball coach has 6 players to assign to 5 different positions (center, power forward, small forward, shooting guard, point guard) on the court, and one will sit on the bench. How many different lineups can the coach make?

(A) 6 (B) 20 (C) 30 (D) 120 **(E) None of the above are true.**

Solution: Note that there are 6 choices for center, and for each of these 5 choices for power forward, and for each of these choices, 4 choices for small forward, etc. Thus, the answer is $(6)(5)(4)(3)(2) = 720$. Therefore the correct answer is "None of the above are true."

20. Which of the following expressions is equivalent to $\dfrac{\sqrt{x^2+1} - x\left(\frac{2x}{2\sqrt{x^2+1}}\right)}{x^2+1}$?

(A) $(2x^2+1)(x^2+1)^{1/2}$ **(B)** $\dfrac{1}{(x^2+1)^{3/2}}$ (C) $(x^2+1)^{1/2}$ (D) $\dfrac{2x^2+1}{(x^2+1)^{3/2}}$

(E) None of the above are true.

Solution:

$$\frac{\sqrt{x^2+1} - x\left(\frac{2x}{2\sqrt{x^2+1}}\right)}{x^2+1} = \frac{(x^2+1)^{1/2} - \frac{x^2}{(x^2+1)^{1/2}}}{x^2+1}$$

$$= \frac{(x^2+1)^{1/2} \cdot \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}} - \frac{x^2}{(x^2+1)^{1/2}}}{x^2+1}$$

$$= \frac{\frac{x^2+1}{(x^2+1)^{1/2}} - \frac{x^2}{(x^2+1)^{1/2}}}{x^2+1}$$

$$= \frac{\frac{1}{(x^2+1)^{1/2}}}{x^2+1}$$

$$= \frac{1}{(x^2+1)^{3/2}}$$

21. What is the digit in the ones place of 2^{57}?

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of the answers (A)–(D) is correct.

Solution: When observing the pattern $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, ..., we see that $2, 4, 8, 6$ repeat in the ones place in cycles of 4. Note that $57 = 4(14) + 1$, so after 14 full cycles of these four values, we have the first element of the next cycle, which is 2. So the ones digit of 2^{57} is 2.

22. A rectangle has sides of 10 centimeters and 8 centimeters. A second rectangle with sides of 7 centimeters and 5 centimeters overlaps the first rectangle such that the overlapping section forms a square. What is the absolute value of the difference between the two non-overlapping regions of the rectangles in square centimeters?

(A) 0 (B) 35 **(C) 45** (D) 80 (E) None of the answers (A)–(D) is correct.

Solution: Denote the length of the overlapping square section by x. Then the area of the square is x^2. The area of the larger rectangle's non-overlapping area is $8(10) - x^2$, and the area of the smaller rectangle's non-overlapping area is $5(7) - x^2$, so that the absolute value of the difference is equal to $[8(10) - x^2] - [5(7) - x^2] = 45$. (Note that the area does not depend on the size of the square.)

23. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

(A) 10/11 (B) 9/10 (C) 8/9 (D) 7/8 (E) None of the answers (A)–(D) is correct.

Solution: After removing x from 10, and then increasing that amount by 10%, we must end up with at least the amount we started with, 10 pounds. That is, the maximum values of x must satisfy $\frac{11}{10}(10 - x) = 10$. Solving for x, we get that $x = 10/11$.

24. An airplane flies due north from Ft. Myers to Sarasota, a distance of 150 miles, and then turns through an angle of 50° and flies to Orlando, a distance of 100 miles. How far is it directly from Ft. Myers to Orlando?

(A) $\sqrt{32500 - 30000\cos(130^\circ)}$ (B) $\sqrt{32500 - 30000\sin(50^\circ)}$ (C) $\sqrt{32500 - 30000\cos(50^\circ)}$

(D) $\sqrt{32500 - 30000\sin(130^\circ)}$ (E) None of the answers (A)–(D) is correct.

Solution: The line segments connecting the three cites form a triangle, two of whose lengths are $a = 150$ miles and $b = 100$ miles. Let c be the third length, which is the distance from Ft. Myers to Orlando. The interior angle at Sarasota is $180^\circ - 50^\circ = 130^\circ$. By the Law of Cosines, we have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(C) \\ &= 150^2 + 100^2 - 2(150)(100)\cos(130^\circ) \\ &= 22500 + 10000 - 30000\cos(130^\circ) \\ &= 32500 - 30000\cos(130^\circ) \end{aligned}$$

Therefore $c = \sqrt{32500 - 30000\cos(130^\circ)}$.

25. Suppose $2x + 10 = 3$ and $4x^2 - 25y^2 = 24$. Which of the following could be a value of $2x + y$?

(A) -6 (B) $-189/5$ (C) 0 (D) 54 (E) None of the answers (A)–(D) is correct.

Solution: Solving the first equation for x, we have $x = -\frac{7}{2}$. Then:

$$\begin{aligned} 4x^2 - 25y^2 &= 24 \\ 4\left(-\frac{7}{2}\right)^2 - 25y^2 &= 24 \\ 4\left(\frac{49}{4}\right) - 25y^2 &= 24 \\ 49 - 25y^2 &= 24 \\ 25y^2 &= 25 \\ y^2 &= 1 \end{aligned}$$

Hence $y = 1$ or $y = -1$. Therefore the two possible values of $2x + y$ are

$$2\left(-\frac{7}{2}\right) + 1 = -7 + 1 = -6$$

and

$$2\left(-\frac{7}{2}\right) - 1 = -7 - 1 = -8.$$

26. Find the equation of the line perpendicular to the line $2x + 4y = -3$ that passes through the center of the circle $x^2 + 6x + y^2 - 8y = 36$.

(A) $y = -\frac{1}{2}x - \frac{5}{2}$ **(B)** $y = 2x + 10$ (C) $y = 2x - 10$ (D) $y = -\frac{1}{2}x + \frac{5}{2}$

(E) None of the answers (A)–(D) is correct.

Solution: Solving the equation of the line for y we see that $y = (-1/2)x - 3/2$, so the line that is perpendicular has slope 2. The equation of the circle can be found by completing the square in both x and y, which yields $x^2 + 6x + 9 + y^2 - 8y + 16 = 61$, after adding 25 to both sides. The left-hand side can be written as $(x+3)^2 + (y-4)^2$ which indicates that the center of the circle is at the point $(-3, 4)$. The desired line has equation $y - 4 = 2(x + 3)$, or $y = 2x + 10$.

27. One of Jason and Jerry always lies on Mondays, Tuesdays and Wednesdays, and always tells the truth on the other days of the week. The other always lies on Thursdays, Fridays, and Saturdays, and always tells the truth the other days of the week. At noon, the two had the following conversation:

Jerry: I lie on Saturdays.

Jason: I will lie tomorrow.

Jerry: I lie on Sundays.

This conversation takes place on a:

(A) Monday (B) Tuesday **(C) Wednesday** (D) Thursday (E) Friday

Solution: Since no one lies on Sundays, Jerry is lying on the day that this conversation takes place (say, today). Since Jerry is lying today, he is lying about lying on Saturdays, so he DOES NOT lie on Saturdays. If Jerry does not lie on Saturdays, then he lies on Mondays, Tuesdays and Wednesdays and, furthermore, today is one of Monday, Tuesday or Wednesday, since he is lying today. Thus, Jason lies on Thursdays, Fridays and Saturdays. Since Jason lies tomorrow and today is Monday, Tuesday or Wednesday, today must be Wednesday.

28. The ratio of the dimensions of a rectangular solid is 3:3:1, and the ratio of its volume to surface area is 3:1. Find the volume of the solid given that the dimensions are measured in meters.

(A) $9000\,\mathrm{m}^3$ (B) $243000\,\mathrm{m}^3$ (C) $27000\,\mathrm{m}^3$ (D) $3000\,\mathrm{m}^3$

(E) None of the answers (A)–(D) is correct.

> **Solution:** The surface area is equal to $2(3x)^2 + 4(x)(3x) = 18x^2 + 12x^2 = 30x^2$, and the volume is equal to $(x)(3x)(3x) = 9x^3$. Since the volume is in a $3:1$ ratio with the surface area, we have $\frac{3}{1} = \frac{9x}{30}$, or $x = 10$. Thus, the volume is $9(10)^3 = 9000$ cubic meters.

29. I want to triple the volume in a cylindrical can. Due to packaging limitations, I can only increase the height of the can by 50%. Assuming I do this, approximate the percentage that the radius of the can must be increased to triple the volume.

    (A) 41.4% (B) 45.7% (C) 52.1% (D) 141.4% (E) None of the answers (A)–(D) is correct.

> **Solution:** Let r and h be the radius and height of the original cyclinder, respectively. Then its volume is $V = \pi r^2 h$. Let R be a new radius such that the volume of the new cylinder is $3V = \pi R^2(1.5h)$. Then
> $$\frac{3V}{V} = \frac{\pi R^2(1.5h)}{\pi r^2 h},$$
> which reduces to $\frac{R}{r} = \sqrt{2} \approx 1.414$, so the radius must be increased by approximately 141.4%.

30. Suppose you are driving a car with 20 inch diameter tires and are traveling down the turnpike at a constant speed of 30π mph. If you have been traveling under these conditions for 30 minutes, how many complete revolutions did your tires make?

    (A) $15,840$ (B) $31,680$ **(C)** $47,520$ (D) $63,360$ (E) None of the answers (A)–(D) is correct.

> **Solution:** The perimeter of the tire is 20π inches. In 30 minutes, the car has traveled 15π miles. There are 5280 feet or $5280(12) = 63360$ inches in one mile. The number of revolutions in 15π miles is then $\frac{63360(15\pi)}{20\pi} = 47520$.

31. Suppose you start with a right triangle, and then you increase its base by 5 inches and decrease its height by 7 inches. If the new shape is an isosceles right triangle with the same area as the original, what is the area (in square inches)?

    (A) 153.125 (B) 167.500 (C) 306.250 (D) 335.001

    (E) None of the answers (A)–(D) is correct.

> **Solution:** Let b and h be the base and height, respectively, of the original triangle. Then the new triangle has base $b + 5$ and height $h - 7$. Since the two triangles have the same area, we know that $\frac{1}{2}bh = \frac{1}{2}(b+5)(h-7)$. Since the new triangle is isosceles and right, we know that $h - 7 = b + 5$. Solving these equations for b and h yields $b = 12.5$ and $h = 24.5$. Hence the area of the first (and also the second) triangle is $\frac{1}{2}(12.5)(24.5) = 153.125$ square inches.

32. A motorboat can maintain a constant speed of 16 miles per hour relative to the water. The boat makes a trip upstream to a certain point in 20 minutes; the return trip takes 15 minutes. What is the speed of the current?

(A) 1.7 mph **(B)** 2.3 mph (C) −1.7 mph (D) 0.6 mph

(E) None of the answers (A)–(D) is correct.

Solution: Let s be the speed of the current. Then the total speed of the motorboat when going upstream is $16 - s$ miles per hour, so the distance traveled is $(16-s)\cdot\frac{20}{60}$ miles. The total speed of the motorboat when going downstream is $16 + s$ miles per hour, so the distance traveled is $(16+s)\cdot\frac{15}{60}$. Since these distances are the same, we have:

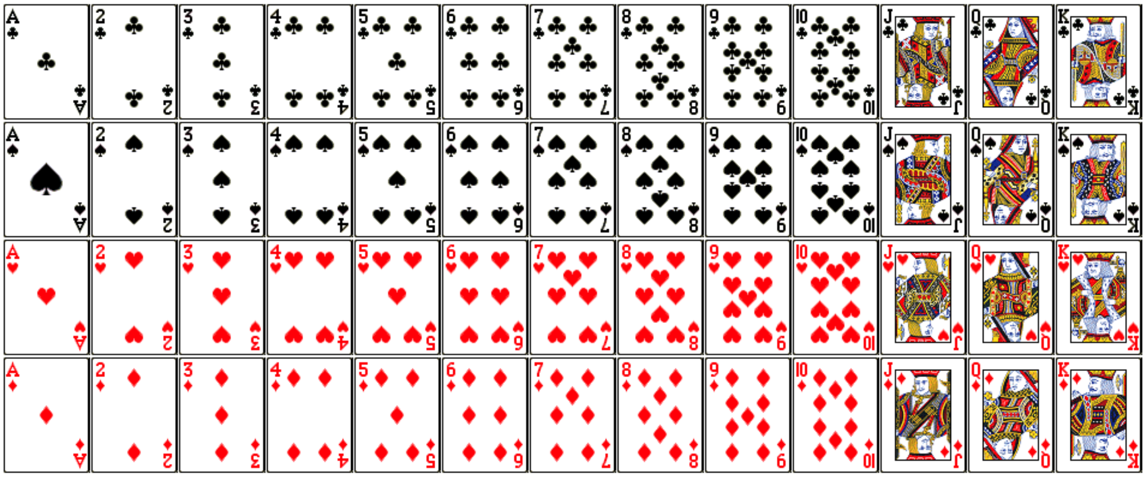
$$\begin{aligned}(16-s)\cdot\frac{20}{60} &= (16+s)\cdot\frac{15}{60}\\ 320 - 20s &= 240 + 15s\\ 80 &= 35s\\ 2.3 &= s\end{aligned}$$

33. Given that $f(x) + 2f(4-x) = x + 8$, compute $f(16)$.

(A) $-32/3$ (B) 9/10 (C) 8/9 (D) 7/8 (E) None of the answers (A)–(D) is correct.

Solution: Plugging in $x = -12$ and $x = 16$, we get that $f(16)+2f(-12) = 24$ and $f(-12)+2f(16) = -4$. Solving for $f(16)$ yields $f(16) = -\frac{32}{3}$.

34. A deck of playing cards contains 52 cards, of which 26 are black and 26 are red. The 26 black cards are divided into two suits of 13 spades and 13 clubs. The 26 red cards are divided into two suits of 13 hearts and 13 diamonds. Each of the suits has cards of ranks Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The ranks of Jack, Queen and King are considered face cards. Let F denote the set of face cards, let D denote the set of diamonds, let R denote the set of red cards and E the set of even-numbered ranks (2, 4, 6, 8, 10). The image below shows each of the cards in a standard deck. Which of the following statements are true?



(A) F and D are independent. (B) R and D are independent. (C) E and F are independent.

(D) Exactly two of the above are true. (E) None of the above are true.

Solution: Recall that two events A and B are **independent** if any of the following are true: $P(A|B) = P(A)$, $P(B|A) = P(B)$, $P(A \cap B) = P(A)P(B)$.

We first consider $P(F|D) = P(F \cap D)/P(D) = (3/52)/(13/52) = 3/13$ and compare this result with $P(F) = 12/52 = 3/13$. Since these results are equal, F and D are independent events. (Logically, this is true since there are the same number of face cards in every suit (3), and the same number of cards in each suit (13), so restricting the cards to be selected only from diamonds does not change the probability of drawing a face card.)

To see if R and D are independent, we consider $P(D|R) = P(D \cap R)/P(R) = (13/52)/(26/52) = 1/2$ and compare this result with $P(D) = 13/52 = 1/4$ and since these results are not equal, we R and D are not independent. (Logically, this is true since half of the red cards are diamonds, but only 1/4 of all cards are diamonds, so, given that a red card is selected, it is more likely that a diamond is selected than if we selected one card from the entire deck.)

Finally, we consider $P(E|F) = P(E \cap F)/P(F) = (0/52)/(12/52) = 0$ and compare this result with $P(E) = 12/52$, and since these are not equal, E and F are not independent. (Logically, this is true since no cards have an even number or a face card, so if a face card is selected, it is impossible that the card has an even number on it.) Thus, only events F and D are independent.