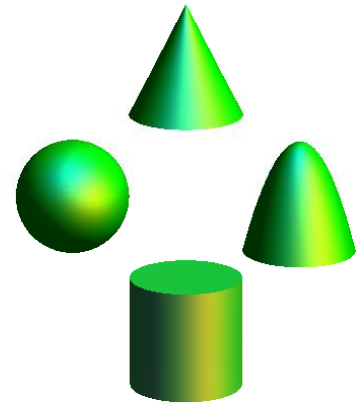
2014
High School Math Contest

Level II
Math 2 & Math 3
Key



Lenoir-Rhyne University

Donald and Helen Schort School of Mathematics and Computing Sciences

This exam is used for both the **Math 2** and the **Math 3** LR Math Contest. **Math 2** students compete against **Math 2** students and **Math 3** students compete against **Math 3** students.

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1. Find the equation of the line perpendicular to the line $3x - 2y = 4$ that passes through the vertex of the parabola $y = x^2 + 6x + 16$.

(A) $y = \dfrac{2}{3}x + 9$ (B) $y = -\dfrac{3}{2}x + \frac{5}{2}$ **(C)** $y = -\dfrac{2}{3}x + 5$ (D) $y = \dfrac{3}{2}x + \dfrac{23}{2}$ (E) $y = -\dfrac{2}{3}x + 7$

Solution: We first find the slope of the line by writing it in $y = mx + b$ form:

$$3x - 2y = 4 \longrightarrow -2y = -3x + 4 \longrightarrow y = \frac{3}{2}x - 3$$

Since $m = \frac{3}{2}$, a perpendicular line will have slope $m_{perp} = -\frac{2}{3}$.

This line must go through the vertex of the parabola. Writing the parabola in $y = a(x-h)^2 + k$ form,

$$y = x^2 + 6x + 16 = (x^2 + 6x + 9) - 9 + 16 = (x+3)^2 + 7,$$

we see that this parabola is the standard parabola $y = x^2$ shifted left 3 units and up 7 units, so its vertex is $(-3, 7)$.

Using point slope form of a line, we have

$$\begin{aligned} y - y_0 &= m(x - x_0), \\ y - 7 &= -\frac{2}{3}(x+3), \\ y - 7 &= -\frac{2}{3}x - 2, \\ y &= \frac{2}{3}x + 5, \end{aligned}$$

so that the solution is **(C)**.

2. Solve $\dfrac{2x}{x^2 - 4} = \dfrac{4}{x^2 - 4} - \dfrac{3}{x+2}$ for x.

(A) 2 (B) $-\dfrac{2}{5}$ (C) -2 (D) $\dfrac{2}{5}$ **(E) None of the answers (A)–(D)**

Solution: Since $x^2 - 4 = (x-2)(x+2)$, if we multiply both sides of the given equation by $x^2 - 4$, we obtain

$$\begin{aligned} 2x &= 4 - 3(x-2), \\ 2x &= 4 - 3x + 6, \\ 5x &= 10, \\ x &= 2. \end{aligned}$$

However, $x = 2$ cannot be a solution to the equation, because it would give 0 in two of the denominators. Thus, the solution is **(E)**, none of the answers (A)–(D).

3. An urn contains 3 red balls, 2 blue balls, and 5 green balls. Two balls are removed at random. What is the probability that one red ball and one green ball are removed?

(A) $\frac{1}{5}$ (B) $\frac{1}{9}$ (C) $\frac{2}{3}$ (D) $\frac{1}{6}$ **(E)** $\frac{1}{3}$

Solution: Note that order doesn't matter. The probability of choosing a red ball first and a green ball second is $\frac{3}{10} \times \frac{5}{9} = \frac{1}{6}$. The probability of choosing a green ball first and a red ball second is $\frac{5}{10} \times \frac{3}{9} = \frac{1}{6}$. Hence, the probability of choosing one red ball and one green ball in either order is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Therefore the correct answer is **(E)**.

4. A circle defined by the equation $x^2 + y^2 = 3$ is intersected by the line $2x - y = 2$. What is the product of the two x-coordinates of the points of intersection?

(A) $\frac{8}{5}$ (B) $-\frac{1}{5}$ **(C)** $\frac{1}{5}$ (D) $-\frac{8}{5}$ (E) None of the answers (A)–(D)

Solution: If we solve the line's equation for y, we get $y = 2x - 2$. Substituting this into the circle's equation, we have

$$\begin{aligned} x^2 + y^2 &= 3, \\ x^2 + (2x-2)^2 &= 3, \\ x^2 + 4x^2 - 8x + 4 &= 3, \\ 5x^2 - 8x + 1 &= 0. \end{aligned}$$

Using the quadratic formula, we deduce

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{(-8)^2 - 4(5)(1)}}{10} = \frac{4 \pm \sqrt{11}}{5}.$$

The product of these x-coordinates is

$$\left(\frac{4 + \sqrt{11}}{5}\right)\left(\frac{4 - \sqrt{11}}{5}\right) = \frac{4^2 - (\sqrt{11})^2}{25} = \frac{16 - 11}{25} = \frac{5}{25} = \frac{1}{5}.$$

Therefore the answer is **(C)**.

5. You have a couch that is 6.5 feet long. You are deciding where you want to place the couch in the living room. You do not want to push the couch around, and you do not have any measuring tools. However, you have several dollar bills in your wallet, and you remember that a dollar bill is 6 inches long. How many dollar bills will it take to measure out the length of the couch?

   (A) 13 dollar bills (B) 26 dollar bills (C) 13 inches (D) 26 inches

   (E) None of the answers (A)–(D)

> **Solution:** We have
>
> $$6.5 \text{ feet} \times \frac{12 \text{ inches}}{1 \text{ foot}} \times \frac{1 \text{ dollar bill}}{6 \text{ inches}} = 13 \text{ dollar bills},$$
>
> so the answer is **(A)**.

6. Calculate the product of the solutions to the equation $x^2 - 6x = -13$.

   (A) 5 **(B) 13** (C) 10 (D) 2 (E) None of the answers (A)–(D)

> **Solution:** In standard form, this is the quadratic equation $x^2 - 6x + 13 = 0$. For such an equation with lead coefficient 1, the product of the two solutions is equal to the constant term of the equation, which is 13. Therefore the answer is **13**. (To see why, suppose the solutions are p and q. Then this equation must factor as $(x - p)(x - q)$, and $(x - p)(x - q) = x^2 - (p + q)x + pq$.)

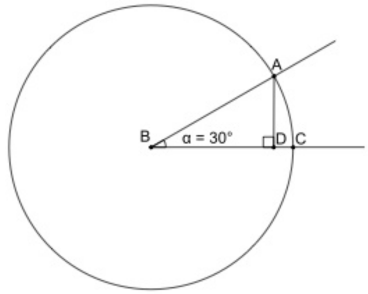
7. $\dfrac{[(4 + 2i)(3 + i) - (5 + 2i)(5 - 2i)]i}{2 + i}$ is equivalent to:

   (A) $-\dfrac{39}{5} - \dfrac{28}{5}i$ (B) $-\dfrac{31}{5} - \dfrac{12}{5}i$ (C) $\dfrac{13}{5} - \dfrac{17}{5}i$ (D) $-\dfrac{1}{5} - \dfrac{48}{5}i$

   (E) None of the answers (A)–(D)

> **Solution:** We compute
>
> $$\begin{aligned}\frac{[(4 + 2i)(3 + i) - (5 + 2i)(5 - 2i)]i}{2 + i} &= \frac{[(12 + 10i - 2) - (25 + 4)]i}{2 + i} \\ &= \frac{[10 + 10i - 29]i}{2 + i} \\ &= \frac{(-10 - 19i)(2 - i)}{(2 + i)(2 - i)} \\ &= \frac{-20 - 38i + 10i - 19}{5} \\ &= \frac{-39 - 28i}{5} \\ &= -\frac{39}{5} - \frac{28}{5}i.\end{aligned}$$
>
> So the answer is **(A)**.

8. Arc AC is subtended by a central angle of $30°$, $\overline{AD}$, which is of length 3, is perpendicular to $\overline{BC}$, and $\overline{BD}$ is of length $3\sqrt{3}$ as seen in the figure below. Find the length of arc AC.



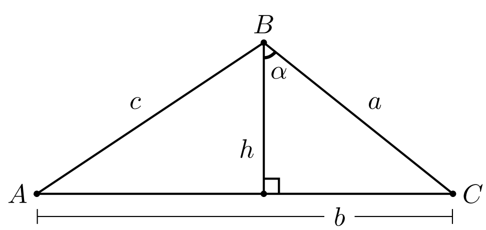
(A) π (B) 6π (C) $\frac{\pi}{6}$ (D) 2π (E) None of the answers (A) through (D)

Solution: The arclength from A to C is equal to the radius times the central angle measured in radians. The radius is the length of segment $\overline{AB}$, which is the hypotenuse of a right triangle with legs $\overline{BD}$ and $\overline{AD}$. Therefore the radius r is

$$r = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9 + 9 \cdot 3} = \sqrt{36} = 6.$$

Since $30°$ is $30 \times \frac{\pi}{180} = \frac{\pi}{6}$ radians, the arclength is $6 \times \frac{\pi}{6} = \pi$. Therefore the answer is **(A)**.

9. Find the area of the triangle shown below. Assume α is measured in degrees.



(A) Area $= \frac{1}{2}ab\sin(\alpha)$ (B) Area $= \frac{1}{2}bc\sin(90° - \alpha)$ **(C) Area $= \frac{1}{2}ab\sin(90° - \alpha)$**

(D) Area $= \frac{1}{2}ac\sin(\alpha)$ (E) None of the answers (A)–(D)

Solution: The triangle containing the angle labeled α is a right triangle. This means that the angle at C must be $90° - \alpha$. Since the side of this triangle opposite C has length h and the hypotenuse has length a, we know that

$$\sin(90° - \alpha) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{a}.$$

Hence $h = a\sin(90° - \alpha)$. Since the large triangle has height h and base b, its area is

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}b \cdot a\sin(90° - \alpha) = \frac{1}{2}ab\sin(90° - \alpha).$$

Therefore the answer is **(C)**.

10. Suppose there are four points such that point A is at $(2, 3)$, point B at $(-4, -3)$, point C at $(4, 5)$, and point D at $(-2, 1)$. Calculate the distance between the midpoints of $\overline{AB}$ and $\overline{CD}$

(A) 13 (B) $\sqrt{61}$ (C) 61 **(D)** $\sqrt{13}$ (E) None of the answers (A)–(D)

Solution: The midpoint of a line segment has coordinates that are the averages of the coordinates of the endpoints, so

$$\text{Midpoint of } \overline{AB} = \left(\frac{2-4}{2}, \frac{3-3}{2}\right) = (-1, 0),$$
$$\text{Midpoint of } \overline{CD} = \left(\frac{4-2}{2}, \frac{5+1}{2}\right) = (1, 3).$$

The distance between these midpoints is

$$\sqrt{(-1-1)^2 + (0-3)^2} = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}.$$

Therefore the answer is **(D)**.

11. Suppose that the number of bacteria in a bottle doubles every 5 minutes. Furthermore, suppose that a single bacterium was present at 6:00 a.m., and the bottle becomes full at 9:00 a.m. What fraction of the bottle is full at 7:15?

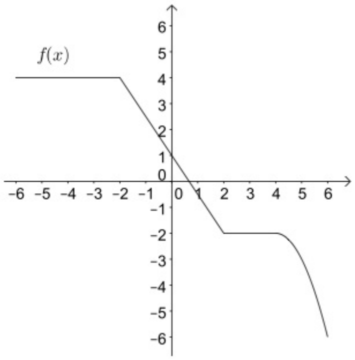
(A) $\frac{1}{2^{105}}$ full (B) 2^{75} full (C) 2^{15} full **(D)** $\frac{1}{2^{21}}$ **full** (E) None of the answers (A)–(D)

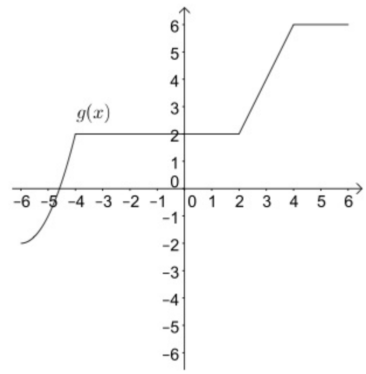
Solution: At 7:15, we know that 75 minutes have passed. We also know that it takes the bacteria 5 minutes to double. Thus there have been $\frac{75}{5} = 15$ doublings that have occurred. Therefore there are 2^{15} bacteria in the bottle at 7:15.

The bottle becomes full at 9:00, and at this time 180 minutes will have passed. We also know that it takes the bacteria 5 minutes to double. Thus there have been $\frac{180}{5} = 36$ doublings that have occurred. Therefore there the full bottle contains 2^{36} bacteria.

At 7:15, there are 2^{15} bacteria out of a total possible of 2^{36} bacteria for a full bottle. Hence the fraction of the bottle that is full at 7:15 is $\frac{2^{15}}{2^{36}} = \frac{1}{2^{21}}$. Therefore the answer is **(D)**.

12. Use the two graphs below to calculate $f(g(3)) + g(f(-4))$.





(A) 0 (B) -2 (C) 8 (D) 6 **(E) None of the answers (A)–(D)**

Solution: We compute

$$\begin{aligned} f(g(3)) + g(f(-4)) &= f(4) + g(4) \\ &= -2 + 6 \\ &= 4. \end{aligned}$$

Since none of the provided answers is correct, the final answer is **(E)**, none of the answers (A)–(D).

13. For what values of k will the product of all solutions to $x^2 + kx + 9 = 0$ be nonnegative?

(A) $k \geq 6$ (B) $k \leq -6$ and $k \geq 6$ (C) $k \geq \frac{9}{2}$ (D) $k \leq -\frac{9}{2}$ and $k \geq \frac{9}{2}$ **(E) all real k**

Solution: Recall that for a quadratic equation $x^2 + bx + c = 0$ with lead coefficient 1, the product of the solutions is equal to the constant term c. (See the solution to problem 6.) In this case $c = 9$, no matter what value k has, so the product of the solutions is nonnegative for all real values of k. Therefore the answer is **(E)**.

14. A polynomial with real coefficients and zeros 3, -1, and $1 + 2i$ is:

(A) $x^4 - 2x^2 + 16x - 15$ **(B)** $x^4 - 4x^3 + 6x^2 - 4x - 15$ (C) $x^4 - 6x^2 - 8x - 3$

(D) $x^4 - 2x^3 + x^2 - 8x - 12$ (E) $x^3 - 7x - 6$

Solution: Since the polynomial has real coefficients and $1 + 2i$ is a root, then its conjugate $1 - 2i$ is also a root. We compute

$$\begin{aligned} (x-3)(x-1)(x-(1+2i))(x-(1-2i)) &= (x-3)(x+1)(x^2-2x+5) \\ &= (x-3)(x^3-x^2+3x+5) \\ &= x^4 - 4x^3 + 6x^2 - 4x - 15. \end{aligned}$$

Therefore the answer is **(B)**.

15. If $x+2$ is a factor of $x^4+x^3+kx^2+5x-2k$, what is k?

(A) 1 (B) -1 (C) 17 (D) -17 (E) None of the answers (A)–(D)

> **Solution:** Recall that $x+2$ is a factor of a polynomial if and only if $x=-2$ is a root of that polynomial. If we substitute $x=-2$ into the given polynomial, we obtain
>
> $$\begin{aligned} x^4+x^3+kx^2+5x-2k &= (-2)^4+(-2)^3+k(-2)^2+5(-2)-2k \\ &= 16-8+4k-10-2k \\ &= 2k-2, \end{aligned}$$
>
> and $2k-2=0$ if and only if $k=1$. Therefore the answer is **(A)**.

16. Which of the following statements is/are true regarding the graph of $f(x)=\dfrac{2x^2-2x-12}{3x^2-27}$?

(A) There is a vertical asymptote at $x=3$. (B) There is a vertical asymptote at $x=-3$.

(C) There is a horizontal asymptote at $y=\dfrac{2}{3}$. **(D) Exactly two of the previous are true.**

(E) Each of the statements in (A)–(C) is true.

> **Solution:** Since
>
> $$f(x)=\frac{2x^2-2x-12}{3x^2-27}=\frac{2(x-3)(x+2)}{3(x-3)(x+3)},$$
>
> there is a vertical asymptote at $x=-3$, a hole in the graph at $x=3$, and a horizontal asymptote of $y=\frac{2}{3}$. Therefore exactly two of the statements (A)–(C) are true, so the answer is **(D)**.

17. Solve for x: $\log_3(\log_2(x))=2$

(A) 3 (B) 9 (C) 81 **(D) 512** (E) 1024

> **Solution:** Unraveling the equation, we have
>
> $$\log_3(\log_2(x))=2 \iff 3^2=\log_2(x) \iff 9=\log_2(x) \iff 2^9=x.$$
>
> Since $2^9=512$, the answer is **(D)**.

18. Solve for x: $\quad 9^{3x-4} = \dfrac{81^{2x+1}}{\sqrt{27}}$.

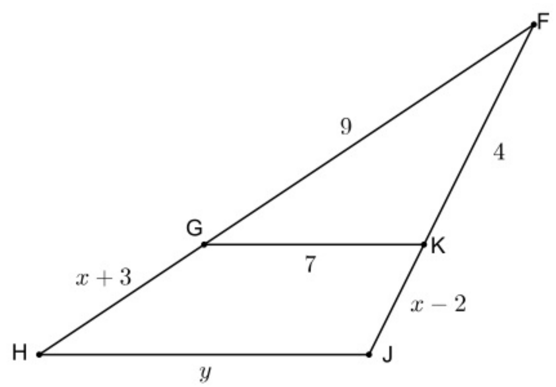
(A) $\frac{14}{5}$ **(B) $-\frac{21}{4}$** (C) $-\frac{7}{3}$ (D) $-\frac{20}{3}$ (E) None of the answers (A)–(D)

Solution: The key is to get every term in the common base 3, and then use the properties of exponentials.

$$\begin{aligned} 9^{3x-4} &= \frac{81^{2x+1}}{\sqrt{27}}, \\ (3^2)^{3x-4} &= \frac{(3^4)^{2x+1}}{3^{3/2}}, \\ 3^{6x-8} &= 3^{8x+4-3/2}, \\ 6x-8 &= 8x+\frac{5}{2}, \\ -2x &= \frac{21}{2}, \\ x &= -\frac{21}{4}. \end{aligned}$$

Therefore the answer is **(B)**.

19. Given $\overline{GK} || \overline{HJ}$ with lengths as shown, find the perimeter of $\triangle HJF$.



(A) 40 (B) 38 (C) 35 (D) 49 (E) 50

Solution: The triangles $\triangle HJF$ and $\triangle GKF$ are similar, so the ratios of their side lengths are equal. Hence we have $\frac{FG}{FK} = \frac{FH}{FJ}$, so $\frac{9}{4} = \frac{9+x+3}{4+x-2} = \frac{x+12}{x+2}$. Cross-multiplying, this means $9(x+2) = 4(x+12)$, so $9x+18 = 4x+48$, so $5x = 30$, so $x = 6$. Also $\frac{FG}{GK} = \frac{FH}{HJ}$, so $\frac{9}{7} = \frac{9+x+3}{y} = \frac{x+12}{y}$. Substituting $x = 6$, this means $\frac{9}{7} = \frac{9+6+3}{y} = \frac{18}{y}$. Cross-multiplying yields that $y = 14$.

The perimeter of $\triangle HJF$ is given by

$$\begin{aligned}\text{Perimeter} &= \overline{FG} + \overline{GH} + \overline{HJ} + \overline{JK} + \overline{KF} \\ &= 9 + x + 3 + y + x - 2 + 4 \\ &= 9 + 6 + 3 + 14 + 6 - 2 + 4 \\ &= 40.\end{aligned}$$

Therefore the answer is **(A)**.

20. Town A and Town B are 200 miles apart. Car A leaves Town A at noon, and drives toward Town B at 40 mph. At 1pm, Car B leaves Town B at 60 mph, driving along the same road toward Town A. After driving 40 miles, Car B realizes they forgot a few things and drove back to Town B. After spending 10 minutes there, Car B gets back on the highway driving toward Town A, again at 60 mph. At what time do the cars meet?

(A) 2:30 pm (B) 3:00 pm **(C) 3:30 pm** (D) 4:00 pm (E) 4:30 pm

Solution: This is a twist on the "standard" car problem. Car B drives 40 miles (which takes 40 minutes), then 40 miles back (which takes another 40 minutes), and then spends 10 minutes at home. Car B is back on the road at 2:30 pm. At 2:30 pm, Car A has driven $40 + 40 + 20 = 100$ miles. Therefore this reduces to a "standard" car problem, where they can be considered to be 100 miles apart driving toward each other starting at 2:30 pm. So if t is measured in hours, we want to solve the equation $40t = 100 - 60t$. This becomes $100t = 100$, so $t = 1$. Therefore the cars will meet one hour after 2:30 pm, so the answer is **(C)**.

21. Which of the following are true about the transformation of $f(x)$ to $g(x)$ if $f(x) = x^2$ and $g(x) = 2x^2 + 5x + 3$?

(A) Shifted right by $\frac{5}{4}$ (B) Shifted down by $\frac{1}{8}$ (C) Shifted left by $\frac{5}{4}$

(D) Both (A) and (B) are correct **(E) Both (B) and (C) are correct**

Solution: Since

$$\begin{aligned}2x^2 + 5x + 3 &= 2\left(x^2 + \frac{5}{2}x + \frac{3}{2}\right) \\ &= 2\left(x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 + \frac{3}{2}\right) \\ &= 2\left(\left(x + \frac{5}{4}\right)^2 - \frac{1}{16}\right) \\ &= 2\left(x + \frac{5}{4}\right)^2 - \frac{1}{8},\end{aligned}$$

there is a vertical stretch by a factor of 2, a shift to the left of $\frac{5}{4}$ and a shift down of $\frac{1}{8}$. Therefore the final answer is **(E)**, both (B) and (C) are correct.

22. Consider a rectangular swimming pool 40 feet long and 25 feet wide. The shallow end is 3 feet deep and extends for 6 feet. Then, for 24 feet horizontally, there is a constant slope downwards to the 10 foot-deep end. One gallon of pool paint covers 80 square feet of surface. How many gallons of paint need to be purchased to ensure the entire surface can be painted?

(A) 22 gallons **(B) 24 gallons** (C) 25 gallons (D) 20 gallons

(E) None of the answers (A) through (D)

Solution: Below is one way to obtain the entire surface area within the pool:

$$\begin{aligned} 3 \times 40 \times 2 &= 240 \\ 3 \times 25 &= 75 \\ 10 \times 25 &= 250 \\ 10 \times 25 &= 250 \\ 6 \times 25 &= 150 \\ 25 \times 25 &= 625 \\ 2 \times 7 \times 24 \times \frac{1}{2} &= 168 \\ 6 \times 10 \times 2 &= 120 \\ \text{Sum} &= 1878 \end{aligned}$$

Then the amount of paint required is $\frac{1878}{80} = 23.475$, so 24 gallons of paint must be purchased. Therefore the answer is **(B)**.

23. Jane conducted a survey of her classmates. The survey asked the students if they enjoyed swimming and if they enjoyed hiking. John wishes to create a two-way frequency table to display the information, but Jane only gave John two bits of information:

$$P(\text{Swimming or Hiking}) = \frac{10}{17} \quad \text{and} \quad P(\text{Swimming given Hiking}) = \frac{5}{7}.$$

He wants the table to look like the empty table below:

	Enjoys Swimming	Doesn't Enjoy Swimming
Enjoys Hiking		
Doesn't Enjoy Hiking		

Which of the following tables should John use?

(A)

7	2
3	5

(B)

5	**2**
3	**7**

(C)

5	3
2	7

(D)

7	3
2	5

(E) None of the answers (A)–(D)

Solution: Since $P(\text{Swimming given Hiking}) = \frac{5}{7}$, we know that of the students who enjoy hiking, $\frac{5}{7}$ of these also enjoy swimming. The following table contains information about the possibilities given for the answer.

Answer	Enjoy Hiking	Enjoy Hiking and Swimming	P(Swimming and Hiking)
(A)	9	7	7/9
(B)	7	5	5/7
(C)	8	5	5/8
(D)	10	7	7/10

Therefore the correct answer must be (B) or (E), none of the above. To determine which, we check whether $P(\text{Swimming or Hiking}) = \frac{10}{17}$ for answer (B).

The only people who don't enjoy swimming or hiking are the people who enjoy neither, and in answer (B) there are 7. Hence there are $17 - 7 = 10$ who enjoy swimming or hiking, so the indeed $P(\text{Swimming or Hiking}) = \frac{10}{17}$.

Therefore the correct answer is **(B)**.

24. Define the functions

$$f(x) = 3x + 4 \qquad g(x) = x^2 - 6x + 8$$

$$h(x) = 5 - x \qquad k(x) = \frac{g(x)}{3f^{-1}(x)}.$$

Suppose $x \neq 4$. Which is equivalent to $(h \circ k)(x)$?

(A) $7 - x$ (B) $3 - x$ (C) $\frac{(x-2)(x-4)}{-3(3x+4)}$ (D) $-x^2 + 5x + 1$ (E) $x^2 - 5x - 1$

Solution: Calculating the inverse of $f(x)$ shows that $f^{-1}(x) = \frac{x-4}{3}$, so that $3f^{-1}(x) = x - 4$. Then

$$k(x) = \frac{x^2 - 6x + 8}{x - 4} = \frac{(x-4)(x-2)}{(x-4)} = x - 2,$$

where the final equality is valid since $x \neq 4$. Hence

$$(h \circ k)(x) = h(x-2) = 5 - (x-2) = 5 - x + 2 = 7 - x.$$

Therefore the answer is **(A)**.

25. Suppose $f(x) = \frac{2x+1}{x-9}$ and $g(x) = x^2 - 16$. If $H(x) = f\big(g(x)\big)$, what is the domain of $(H \cdot H)(x)$?

(A) $(-\infty, 5) \cup (5, \infty)$ **(B)** $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$ (C) $(-\infty, -25) \cup (-25, 25) \cup (25, \infty)$

(D) $(\infty, 4) \cup (4, \infty)$ (E) $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

Solution: First consider the domain of $H(x) = f\Big(g(x)\Big)$. We need x in the domain of g, and $g(x)$ in the domain of f. The domain of g is all real x, and the domain of f is $x \neq 9$. So we require $x^2 - 16 \neq 9$, which means $x^2 \neq 25$, which means $x \neq \pm 5$. Since $(H \cdot H)(x) = H(x) \cdot H(x)$, the domain of $H \cdot H$ is the same as the domain of H, so the only restriction is $x \neq \pm 5$. Therefore the correct answer is **(B)**.

26. Suppose we start with the graph of the circle $x^2 + 4x + y^2 + 6y = 12$. We shift the graph left 7 units, then shift up 5 units, then reflect it over the x-axis. Finally, we rotate the circle 90° clockwise about the point $(-4, -2)$. What is the product of the x and y coordinates of the center of the resulting circle?

(A) -12 (B) 12 (C) -18 (D) 18 (E) None of the answers (A)–(D)

Solution: We first complete the squares to find the center of the circle:

$$x^2 + 4x + y^2 + 6y = 12$$
$$(x^2 + 4x + 4) + (y^2 + 6y + 9) = 12 + 4 + 9$$
$$(x + 2)^2 + (y + 3)^2 = 25$$

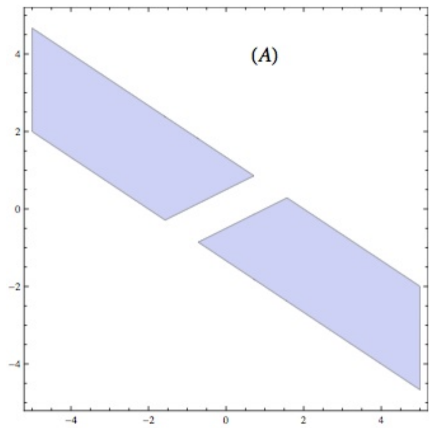
The center of the circle is originally at $(-2, -3)$.

Shifting left by 7 units: $(-9, -3)$
Shifting up by 5 units: $(-9, 2)$
Reflecting over the x-axis: $(-9, -2)$

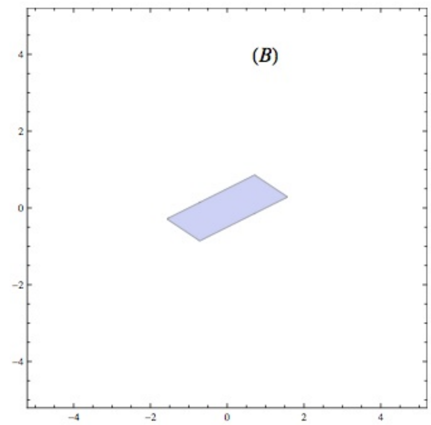
Notice that this point is 5 units left of $(-4, -2)$. When we rotate the circle 90° clockwise about the point $(-4, -2)$, the center will now be 5 units above $(-4, -2)$, that is, at $(-4, 3)$. Therefore the product of the x and y coordinates is -12, so the correct answer is **(A)**.

27. Which shaded region illustrates the solutions to

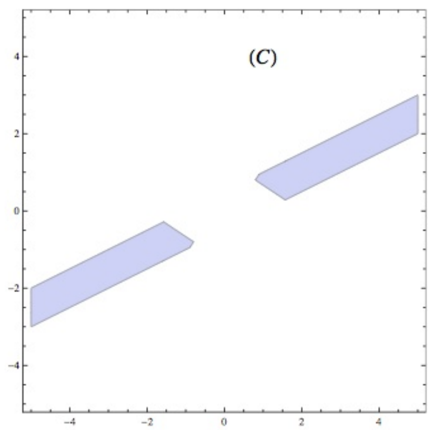
$$\begin{cases} |2x+3y| \le 4 \\ |x-2y| \ge 1 \\ |x| \le 5 \\ |y| \le 5 \end{cases}$$



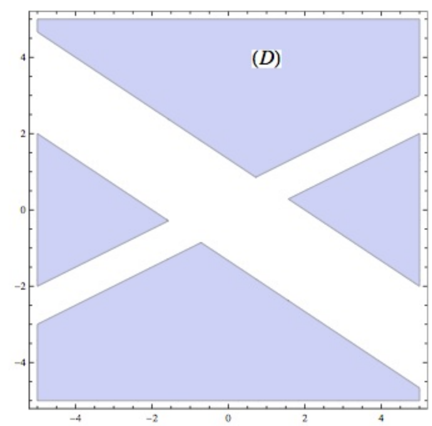
(A)



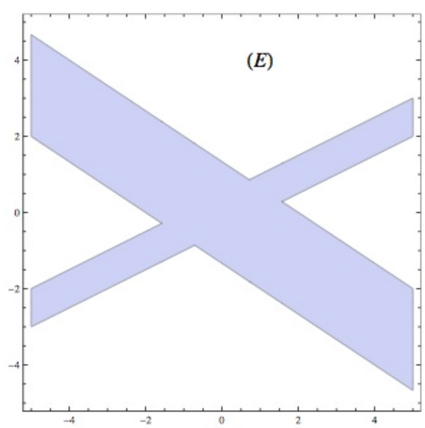
(B)



(C)



(D)



(E)

Solution: The inequality $|2x+3y| \le 4$ is equivalent to $-4 \le 2x+3y \le 4$, and

$$-4 \le 2x+3y \le 4 \iff -2x-4 \le 3y \le -2x+4 \iff -\frac{2}{3}x - \frac{4}{3} \le y \le -\frac{2}{3}x + \frac{4}{3}.$$

The inequality $|x-2y| \ge 1$ is equivalent to the statement $x-2y \ge 1$ or $x-2y \le -1$. Note that

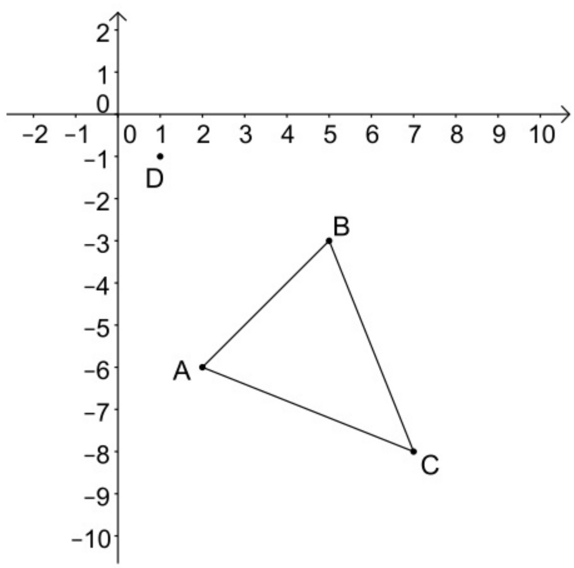
$$x-2y \ge 1 \iff -2y \ge 1-x \iff y \le \frac{1}{2}x - \frac{1}{2}$$

and

$$x-2y \le -1 \iff -2y \le -1-x \iff y \ge \frac{1}{2}x + \frac{1}{2}.$$

The only graph that satisfies all of these requirements is **(A)**.

28. The triangle shown below has vertices A at $(2,-6)$, B at $(5,-3)$, and C at $(7,-8)$. If the center of dilation is point D at $(1,-1)$ and the scale factor is $1/2$, calculate the product of the x-coordinates of vertices A and C after the dilation.



(A) 12 (B) $\frac{9}{2}$ **(C) 6** (D) $\frac{63}{4}$ (E) None of the answers (A)–(D)

Solution: With the center of dilation at $(1,-1)$ and a scale factor of $\frac{1}{2}$, the new vertices will be A at $(\frac{3}{2}, -\frac{7}{2})$, B at $(3,-2)$, and C at $(4,-\frac{9}{2})$. The product of the x-coordinates of vertices A and C is $\frac{3}{2} \times 4 = 6$, so the final answer is **(C)**.

29. Derive the equation of a parabola with focus at $(-3, 4)$ and the directrix $y = 2$.

(A) $y = \frac{1}{4}x^2 + \frac{3}{2}x + \frac{21}{4}$ (B) $y = \frac{1}{4}x^2 - \frac{3}{4}$ (C) $y = \frac{1}{4}x^2 + \frac{21}{4}$ (D) $y = \frac{1}{4}x^2 - \frac{3}{2}x - \frac{3}{4}$

(E) None of the answers (A)–(D)

Solution: For a point (x, y) on this parabola, the distance from (x, y) to $(-3, 4)$ must be the same as the distance from (x, y) to the line $y = 2$. The distance between the two points is

$$\begin{aligned}\sqrt{(x-(-3))^2 + (y-4)^2} &= \sqrt{(x+3)^2 + (y-4)^2} \\ &= \sqrt{x^2 + 6x + 9 + y^2 - 8y + 16} \\ &= \sqrt{x^2 + 6x + y^2 - 8y + 25}.\end{aligned}$$

The distance between (x, y) and the line $y = 2$ is $y - 2$. (Since the focus is above the directrix, we can assume that the point (x, y) is above the directrix also, so y is larger than 2.) Therefore $\sqrt{x^2 + 6x + y^2 - 8y + 25} = y - 2$. Squaring both sides yields:

$$\begin{aligned}x^2 + 6x + y^2 - 8y + 25 &= (y-2)^2, \\ x^2 + 6x + y^2 - 8y + 25 &= y^2 - 4y + 4, \\ x^2 + 6x + 21 &= 4y, \\ \frac{1}{4}x^2 + \frac{3}{2}x + \frac{21}{4} &= y.\end{aligned}$$

Therefore the correct answer is **(A)**.

30. A sample of concrete specimens of a certain type is selected, and the compressive strength of each specimen is determined. The mean and standard deviation are calculated as $\bar{x} = 3000$ and $s = 500$ respectively. The sample histogram is found to be well approximated by a normal curve. Using the empirical rule, approximately what percent of the sample observations are between 2500 and 4000?

(A) 95% **(B)** 81.5% (C) 83.85% (D) 68% (E) None of the answers (A)–(D)

Solution: The z-score for 2500 is

$$z = \frac{2500 - \bar{x}}{s} = \frac{2500 - 3000}{500} = -1$$

and the z-score for 4000 is

$$z = \frac{4000 - \bar{x}}{s} = \frac{400 - 3000}{500} = 2.$$

We need to use the fact that a normal curve is symmetrical together with the empirical rule. Since 2500 is 1 standard deviation below the mean and 4000 is 2 standard deviations above the mean, we need half of the 68% rule and half of the 95% rule. Therefore the final answer is 81.5%, which is **(B)**.