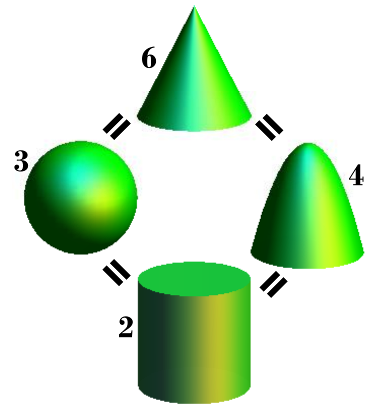
2011

High School Math Contest

Geometry Exam



Lenoir-Rhyne University

Donald and Helen Schort School of
Mathematics and Computing Sciences

Geometry Solutions

This exam has been prepared by the following faculty from **Western Carolina University**:

Risto Atanasov
Kate Best
Mark Budden
Geoff Goehle
Axelle Faughn

HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
WESTERN CAROLINA UNIVERSITY

GEOMETRY ANSWERS
2011

Prepared by:
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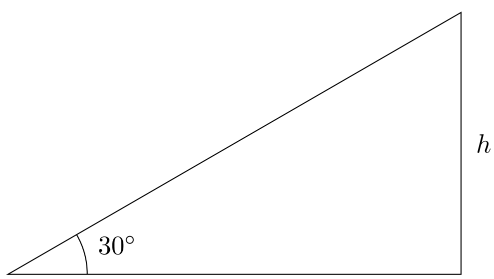
1. (c) $\angle BDC = 40^\circ$
   $\angle ABD = 40^\circ = \angle DAB$
   Hence $\angle ADB = 100^\circ$.

2. (a) Let $\angle BAO = x$. Applying the theorem of external angles, we have $3x = 45^\circ$. So $x = 15^\circ$, i.e. $\angle BAO = 15^\circ$.

3. (a)



$$\tan 30^\circ = \frac{h}{200}$$
$$\tan 30^\circ = \frac{\sqrt{3}}{3}$$
$$\text{So } h = 200\frac{\sqrt{3}}{3}$$

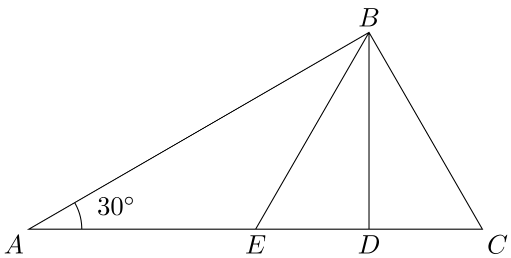
4. (c) Area of circle $= \pi r^2$
   Area of square $= (2r)^2$
   So probability to hit circle $= \frac{\pi r^2}{4r^2} = \frac{\pi}{4}$
   Probability to hit outside circle $= 1 - \frac{\pi}{4}$

5. (a) $(-20, 8) \to (-5, -2)$. Dilation factor with center 0: $\frac{1}{4}$.
   $(-40, -4) \to (-10, -1)$

6. (b)

$$2x^2 = 16$$
$$x^2 = 8$$
$$x = 2\sqrt{2}$$
$$\text{Area } = \frac{1}{2}\left(2\sqrt{2}\right)^2 = \frac{1}{2} \cdot 8 = 4.$$

7. (d)



$\triangle ABC$ is a $30° - 60°-$ right triangle
$\Rightarrow \overline{AC} = 6$ and $\overline{BC} = 3$.
$\triangle BDC$ is also a $30° - 60°-$ right triangle
$\Rightarrow \overline{DC} = \frac{3}{2}$. Also, E is the midpoint of $\overline{AC} \Rightarrow \overline{EC} = 3$. Thus, $\overline{DE} = \frac{3}{2}$.

8. (b) Let the perimeter of the equilateral triangle and hexagon both be equal to $6s$ where s is the length of each side of the hexagon and $2s$ is the length of each side of the triangle. Since the area of the triangle with side length $2s$ is 2 square units, the area of an equilateral triangle with side length s would be $\frac{1}{2}$ sq. unit. There are six such triangles in the hexagon. The area of the hexagon then is 3 square units.

9. (b)

$$(8+3x) + (2x+96) + (92+2x) = 476$$
$$196 + 7x = 476$$
$$7x = 280$$
$$x = 40$$
$$(8+3x) + (2x+96) + (92+2x) = 128 + 176 + 172$$
$$24 + 6x = 264$$
$$\text{similarity ratio } = \frac{264}{176} = 1.5$$
$$\text{So perimeter of } DEF = (476)(1.5) = 714$$

10. (c) $(10 \cdot 8 - a) - (7 \cdot 5 - a) = 80 - 35 = 45$ sq units

11. (d) Using the formula $A = \frac{1}{2}bh$ for a triangle, the region outside the shape can be divided into square regions and triangles with a total area of $10\frac{1}{2}$ sq. units. Since the whole board is 16 sq. units, the area enclosed by the region is $16 - 10\frac{1}{2} = 5\frac{1}{2}$ square units.

12. (b) Since F is the midpoint, the altitude of $\triangle AEF$ from F to AE is one half the altitude of $\triangle ABC$ from C to AB. Base AE of $\triangle AEF$ is $\frac{3}{4}$ of the base AB of $\triangle ABC$. Therefore $\triangle AEF$ is $\frac{1}{2} \cdot \frac{3}{4} \cdot 96 = 36$.

13. (a) The radius of the ball is 5 so the volume of clay is $\frac{4}{3}\pi(5)^3$. The volume of the cylinder is $\frac{4}{3}\pi(5)^3 = \pi r^2(10)$ so the radius of the cylinder is $r = 5\sqrt{\frac{2}{3}} = \frac{5\sqrt{6}}{3}$cm.

14. (b)

$$\begin{aligned} \frac{40}{30} &= \frac{40+56+84}{30+35+x} \\ \frac{4}{3} &= \frac{180}{65+x} \\ 260+4x &= 540 \\ 4x &= 280 \\ x &= 70 \\ \Rightarrow \triangle ABC &= 70+35+30+40+56+84 = 315 \end{aligned}$$

15. (c) $\triangle AFB$ is inscribed in a semicircle $\Rightarrow \angle FAB = 90° \Rightarrow \angle FBA = 55°$.
Supplementary angles $\Rightarrow \angle FBD = 125°$.
Vertical angles $\Rightarrow \angle BDE = 70°$.
Tangency $\Rightarrow \angle DEC = 90°$.
So, $360° = 90° + 70° + 125° + \angle a \Rightarrow \angle a = 75°$.

16. (b) Since $\overline{AC}$ and $\overline{BD}$ are radii, $\overline{CD} = 5$.
Similar triangle gives

$$\begin{aligned} \frac{\overline{CE}-5}{\overline{CE}} = \frac{2}{3} &\Rightarrow 3\overline{CE} - 15 = 2\overline{CE} \\ &\Rightarrow \overline{CE} = 15. \end{aligned}$$

17. (d) The diagonal of the base is found by

$$\begin{aligned} c^2 &= 2^2 + 5^2 = 29 \\ c &= \sqrt{29} \end{aligned}$$

Then d is found by

$$\begin{aligned} d^2 &= 3^2 + d^2 = 9 + 29 = 38 \\ \Rightarrow d &= \sqrt{38} \text{ ft} \end{aligned}$$

18. (d)

$$\frac{500}{3}\pi = \frac{4}{3}\pi r^3 \Rightarrow r^3 = 125 \text{ cm} \Rightarrow r = 5 \text{ cm}.$$

The diagonal of the cube is $2r = 10 = \sqrt{3s^2}$ where s is the side length of the cube. $10 = \sqrt{3}s \Rightarrow s = \frac{10}{\sqrt{3}}$.
So the volume of the cube is $\left(\frac{10}{\sqrt{3}}\right)^3 = \frac{1000\sqrt{3}}{9}$ cm^3.

19. (e) The interior angles of a regular pentagon are $108°$. $360° - 36° = 324°$ for reflex angle a.

20. (c) Since $D = 81$ sq. units and $C = 64$ sq. units, $I = 1$ sq. unit and $D = 81$ sq. units, $E = 100$ sq. units and $H = 49$ sq. units. Therefore G is 16 sq. units and F is 196 sq. units.

21. (d) The volume of all three balls combined can be represented as

$$\begin{aligned} \frac{4}{3}\pi r^3 \cdot 3 = \frac{4}{3}\pi\left(\frac{1}{2}d\right)^3 \cdot 3 = \\ 4\pi\left(\frac{1}{8}d^3\right) = \frac{1}{2}\pi d^3 \end{aligned}$$

The volume of the cylinder can be represented as:

$$\pi r^2 \cdot 3d = \pi \left(\frac{1}{2}d\right)^2 \cdot 3d$$
$$\pi \frac{1}{4}d^2 \cdot 3 = \frac{3}{4}\pi d^3$$

The space taken up by the balls is $\frac{\frac{1}{2}\pi d^3}{\frac{3}{4}\pi d^3} = \frac{2}{3}$ of the space in the cylinder. The unused space then is $\frac{1}{3}$ of the cylinder.

22. (c) The small triangles formed by extending the sides of the octagon are isosceles right triangles with sides $\sqrt{2}$. So the base is $2 + 2\sqrt{2}$ and the height is $2 + \sqrt{2}$. Therefore the area is

$$\begin{aligned}\frac{1}{2}b \cdot h &= \frac{1}{2}\left(2 + 2\sqrt{2}\right)\left(2 + \sqrt{2}\right)\\ &= \frac{1}{2}\left(4 + 2\sqrt{2} + 4\sqrt{2} + 4\right)\\ &= \frac{1}{2}\left(8 + 6\sqrt{2}\right)\\ &= 4 + 3\sqrt{2}\end{aligned}$$

23. (d) Using similar triangles we can break the ladder into two smaller triangles at the corner. The legs of the upper triangle have length 5 and the legs of the lower triangle have length 10. The ladder is then

$$\sqrt{5^2 + 5^2} + \sqrt{10^2 + 10^2} = 15\sqrt{2}$$

This could be made more computationally difficult using worse numbers.

24. (a) The hardest part is finding the height of the triangle formed by the centers of the logs. However that triangle is equilateral with sides 8. So the height is $\sqrt{8^2 - 4^2} = 4\sqrt{3}$. The height of the pile is then

$$8 + 4\sqrt{3}$$

25. (d) You use the fact that the triangles are similar to conclude that (from the law of sines)

$$y = \frac{4\sin\beta}{3\sin\alpha}x$$

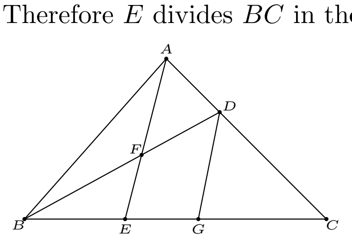
26. (a) Let a, b, and c be the lengths of the sides of the triangle. Then

$$A = \frac{1}{2}(ra + rb + rc) = \frac{1}{2}rP.$$

Hence $\frac{P}{A} = \frac{2}{r}$.

27. (c) Draw DG parallel to AE. Then $\overline{BE} = \overline{EG}$ because $\overline{BF} = \overline{FD}$. Also $2\overline{EG} = \overline{GC}$ because $2\overline{AD} = \overline{DC}$. Hence $3\overline{BE} = \overline{EG} + \overline{GC} = \overline{EC}$.

Therefore E divides BC in the ratio of $1 : 3$.



28. (a) The radius of OB is perpendicular to AC and bisects AC at E. CD is parallel to EO and $\overline{EO} = \frac{1}{2}\overline{CD}$. Since the triangles BEA and ABD are similar we have

$$\frac{\overline{BE}}{1} = \frac{1}{\overline{AD}} = \frac{1}{4}$$
$$\overline{EO} = \overline{BO} - \overline{BE} = \frac{7}{4}$$

Hence $\overline{CD} = 2\overline{EO} = \frac{7}{2}$.

29. (b) Let A_1 and A_2 represent the larger and smaller area respectively. Let x be the length of the side of the larger triangle corresponding to the side of length 3 in the smaller triangle. Then

$$\frac{A_1}{A_2} = \frac{x^2}{3^2} = k^2 \text{ for some positive integer } k$$
$$\frac{A_2 + 18}{A_2} = \frac{x^2}{3^2} = k^2.$$
$$A_2 = \frac{18}{k^2 - 1}$$

Since A_2 is an integer, $k^2 = 4$. Hence $\frac{x^2}{3^2} = 4$, i.e. $x = 6$cm.

30. (b)

$$\frac{A_{\triangle MDC}}{A_{\triangle ADC}} = \frac{1}{k}$$
$$\frac{A_{\triangle NAB}}{A_{\triangle ABC}} = \frac{1}{k}$$
$$\text{So } A_{\triangle MDC} = \frac{1}{k}A_{\triangle ADC}, A_{\triangle NAB} = \frac{1}{k}A_{\triangle ABC}$$
$$A_{\triangle MDC} + A_{\triangle NAB} = \frac{1}{k}A_{ABCD}$$
$$A_{AMCN} = A_{ABCD} - (A_{\triangle MDC} + A_{\triangle NBA}) = \frac{k-1}{k}A_{ABCD}$$

The ratio is $(k-1) : k$.