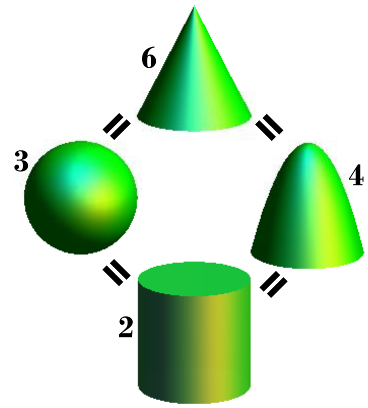
2011

High School Math Contest



Algebra II Exam

Lenoir-Rhyne University

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Algebra II Solutions

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HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
WESTERN CAROLINA UNIVERSITY

ALGEBRA II ANSWERS
2011

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1. (b)

$$\begin{aligned} y &= -3x^2+6x-1 \\ &= -3(x^2-2x+1)-1+3 \\ &= -3(x-1)^2+2 \\ y &= 2 \end{aligned}$$

2. (d)

$$f(x)=\frac{2(x-5)(x+2)}{(x+2)(x+7)}$$

Hole at $x=-2$. Vertical asymptote at $x=-7$.

3. (a) $\frac{(f\circ g)(4)}{(g\circ f)(4)}=\frac{f\left(\frac{3}{4}\right)}{g(2)}=\frac{-\frac{5}{4}}{\frac{3}{2}}=-\frac{5}{6}$

4. (c)

$$\begin{aligned} g(x) &= (x^2-4)(x^2+4) \\ &= (x+2)(x-2)(x^2+4) \\ x &= -2 \;\; x=2 \\ & 2 \end{aligned}$$

5. (e)

$$\begin{cases} 2x+y=-1 \\ -4x-3y=0 \end{cases} \Rightarrow \begin{array}{rrl} 4x & +2y & =-2 \\ -4x & -3y & =0 \\ \hline & -y & =-2 \end{array} \Rightarrow \begin{array}{r} y=2 \\ x=-\frac{3}{2} \end{array} \Rightarrow (-\frac{3}{2},2) \Rightarrow 3\cdot-\frac{3}{2}+2(2)=-\frac{1}{2}$$

6. (b) This cancels to 1.

7. (e)

8. (c) The common ratio is $\frac{24}{16}=\frac{36}{24}=\frac{54}{31}=\frac{81}{54}=1.5$ so this is an exponential function.

9. (d)

$$\ln\left(\frac{7x\sqrt{3-4x}}{2(x-1)^3}\right)$$
$$\ln\left(7x(3-4x)^{\frac{1}{2}}\right) - \ln\left(2(x-1)^3\right)$$
$$\ln 7x + \ln(3-4x)^{\frac{1}{2}} - \ln 2 - \ln(x-1)^3$$
$$\ln 7 + \ln x + \frac{1}{2}\ln(3-4x) - \ln 2 - 3\ln(x-1)$$

10. (b)
Using determinants:

$$\begin{vmatrix} -4 & 2 & 1 \\ -1 & 0 & 1 \\ 3 & 8 & 1 \end{vmatrix} = (-4)\begin{vmatrix} 0 & 1 \\ 8 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ 3 & 8 \end{vmatrix}$$

$$= (-4)(-8) - (2)(-4) + (1)(-8)$$
$$= 32 + 8 - 8 = 32 \text{ sq. miles}$$

Area $= \frac{\det}{2} = 16$ sq. miles.

11. (d)
Center of circle: $\left(\frac{-2+4}{2}, \frac{-1+5}{2}\right) = (1, 2)$
Radius of circle:
$$\frac{\sqrt{(4-(-2))^7 + (5-(-1))^2}}{2} = \frac{\sqrt{6^2+6^2}}{2} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$
Equation of circle in standard form: $(x-1)^2 + (y-2)^2 = 18$

12. (a) This is synthetic division of $-2x^3 + 13x^2 - 17x - 12$ by $x - 3$. You get

$$-2x^2 + 7x + 4$$

which factors as $(2x+1)(4-x)$. So the remaining roots are $-1/2$ and 4. The product of all roots is then $-\frac{1}{2} \cdot 4 \cdot 3 = -6$.

13. (b) $(4, \infty)$

14. (a) Since the area of the larger circle is $A = 9\pi$ the area of the smaller circle is 3π and its radius is $\sqrt{3}$.
For $x = 5$ and $y = 1$, $(5 - (5 - \sqrt{3}))^2 + (1-1)^2 = 3$
$\left(x - \left(5 - \sqrt{3}\right)\right)^2 + (y-1)^2 = 3$

15. (b)

$$y = 3(x-2)^2 + 6$$
$$\text{Shift left 3 units } \to 3(x-2+3)^2 + 6$$
$$\text{Stretch by 5 } \to 5\left(3(x-2+3)^2 + 6\right)$$
$$\text{Shift up 1 } \to 5\left(3(x-2+3)^2\right) + 30 + 1$$
$$\text{So } y = 15(x+1)^2 + 31$$

16. (a)

$$\begin{aligned} x = \frac{\log 3}{\log 2} \cdot \frac{\log 4}{\log 3} \cdot \ldots \cdot \frac{\log 64}{\log 63} = \\ = \frac{\log 64}{\log 2} = \log_2 64 = 6. \end{aligned}$$

17. (a) Since $g(x) = 1$ is satisfied for $x = 1$ or $x = -1$, we have

$$f(1) = f\left(g(1)\right) = \frac{1 - 1^2}{1 + 1^2} = 0.$$

18. (a) By substituting $t = 2^x$ we have

$$\begin{gathered} t^2 - 10t + 16 = 0 \\ (t-2)(t-8) = 0 \\ t = 2 \text{ or } t = 8 \\ 2^x = 2 \text{ or } 2^x = 8 \end{gathered}$$

Hence $x = 1$ or $x = 3$. Then $x^2 + 1$ is 2 or 10.

19. (b) The given equation is equivalent to

$$x^2 - \frac{5m+6}{m+1}x + \frac{6m+5}{m+1} = 0$$

Then $x_1 + x_2 = \frac{5m+6}{m+1}$, $x_1 x_2 = \frac{6m+5}{m+1}$.

$$x_1 + x_2 + x_1 x_2 = \frac{11m + 11}{m+1} = 11.$$

20. (b)

$$\begin{gathered} \frac{(n+2)!}{(n+2) \cdot (n-1)!} = \frac{(n+1)!}{(n-1)!} = \\ n(n+1) = 30 \\ n = 5 \end{gathered}$$

21. (c)

$$\begin{gathered} x - y = -1 \\ y = x + 1 \\ x^2 + (x+1)^2 - 2x - 2(x+1) - 23 = 0 \\ 2x^2 + 1 - 2x - 2 - 23 = 0 \\ 2x^2 - 2x - 24 = 0 \\ 2(x^2 - x - 12) = 0 \\ 2(x+3)(x-4) = 0 \\ x = -3 \quad x = 4 \\ (-3, -2) \text{ and } (4, 5) \end{gathered}$$

22. (b)

$$\begin{aligned}
\ln 9^{2-5x} &= \ln 27 \\
(2-5x)\ln 9 &= \ln 27 \\
2-5x = \frac{\ln 27}{\ln 9} &= \log_9 27 = \frac{3}{2} \\
2-5x &= \frac{3}{2} \\
-5x &= -\frac{1}{2} \\
x &= \frac{1}{10} \\
\log x = \log 10^{-1} &= -1
\end{aligned}$$

23. (b)

$$\begin{aligned}
y &= 4 - 3^{x-2} \\
x &= 4 - 3^{y-2} \\
x - 4 &= -3^{y-2} \\
4 - x &= 3^{y-2} \\
y - 2 &= \log_3(4-x) \\
y &= \log_3(4-x) + 2 \\
h^{-1}(x) &= \log_3(4-x) + 2
\end{aligned}$$

24. (d) The vertex of the parabola is at $x = 6/2 = -3$ so we have the coordinates $(3, -2)$. The slope we are looking for is $-1/2$. The equation then is

$$y = -1/2(x-3) - 2$$

or any of its variants.

25. (c) We need $-2x^2 - 9x + 18 > 0$ or $-(x+6)(2x-3) > 0$. So we need one, but not both of $x + 6 < 0$ and $2x - 3 < 0$. So we need one, but not both of $x < -6$ and $x < 3/2$. So the interval is -6 to $3/2$.

26. (e) Let $N_1 =$ lbs. of the less expensive nuts, $N_2 =$ lbs. of more expensive nuts. So $12 = N_1 + N_2$. Let $p =$ price per lb. of less expensive nuts. Then $30 = N_1(p)$ and $70 = N_2(p+4)$. Substituting:

$$\begin{aligned}
70 &= (12 - N_1)(\frac{30}{N_1} + 4) \\
70 &= \frac{360}{N_1} + 18 - 4N_1 \\
70N_1 &= 360 + 18N_1 - 4{N_1}^2 \\
{N_1}^2 + 13N_1 - 90 &= 0 \\
(N_1 - 5)(N_1 + 18) &= 0 \\
N_1 = 5 \text{ lbs. } N_2 &= 7 \text{ lbs.} \\
7 \text{ lbs. } - 5 \text{ lbs. } &= 2 \text{ lbs.}
\end{aligned}$$

27. (c)

$$x^3 - 3x^2 + 25x - 75$$
$$= x^2(x-3) + 25(x-3)$$
$$(x^2+25)(x-3)$$
$$(x^2+25)(x-3) = 0 \text{ for } x = 3, x = -5i$$

28. (b)

$$s(.07) + e(.08) \geq \$638$$
$$s + e = \$8500 \rightarrow s = 8500 - e$$
$$(8500 - e)(.07) + e(.08) \geq \$638$$
$$8500(\frac{7}{100}) + e(.01) \geq \$638$$
$$\frac{e}{100} \geq \$43$$
$$e \geq \$4300$$

29. (c)

$$x^2 + y^2 - 64 = 0$$
$$x^2 - 5y - 3 = 0$$
$$5y + 3 + y^2 - 64 = 0$$
$$y^2 + 5y - 61 = 0$$
$$y^2 + 2\frac{5}{2}y + \left(\frac{5}{2}\right)^2 - (\frac{5}{2}) - 61 = 0$$
$$\left(y + \frac{5}{2}\right)^2 - \frac{269}{4} = 0$$
$$\left(y + \frac{5}{2} - \frac{\sqrt{269}}{2}\right)\left(y + \frac{5}{2} + \frac{\sqrt{269}}{2}\right) = 0$$
$$y = \frac{\sqrt{269}-5}{2} \rightarrow x = \pm\sqrt{5\frac{\sqrt{269}-5}{2} + 3}$$
$$y = \frac{-\sqrt{269}-5}{2} \rightarrow \text{imaginary solutions}$$

30. (c)

$$x - 8y \geq -8 \rightarrow y \leq 1 + \frac{x}{8}$$
$$5x + 3y \geq 3 \rightarrow y \geq 1 - \frac{5x}{3}$$
$$6x - 5y \leq 3 \rightarrow y \geq -\frac{3}{5} + \frac{6x}{5}$$

$$1+\frac{x}{8}=1-\frac{5x}{3}\Rightarrow(x=0,y=1)$$
$$1+\frac{x}{8}=-\frac{3}{5}+\frac{6x}{5}\Rightarrow\frac{8}{5}=\frac{48x-5x}{40}\Rightarrow 8=\frac{43x}{8}\Rightarrow x=\frac{64}{43}$$
$$1-\frac{5x}{3}=-\frac{3}{5}+\frac{6x}{5}\Rightarrow\frac{8}{5}=\frac{5x}{3}+\frac{6x}{5}=\frac{43x}{15}\Rightarrow x=\frac{24}{43},y=\frac{3}{43}$$

31. (d)
32. (a)

$$(2\cdot 1-1)^{2011}=a_0+a_1+a_2+\ldots+a_{2011}$$
$$\Rightarrow a_0+a_1+a_2+\ldots+a_{2011}=1.$$

33. (b)

$$x^2+ax+1=x^2-x-a$$
$$ax+x+1+a=0$$
$$(a+1)(x+1)=0$$

Hence $a=-1$ or $x=-1$.
If $a=-1$, then the given equations are identical and they have two complex but not real solutions.
If $x=-1$ is a common solution to the given equations, then $a=2$.
Therefore, the equations have a common solution if $a=2$.

34. (b) From $x^2+x+1=0$ we get $x^2+1=-x$. Also $x^3-1=(x-1)(x^2+x+1)=0$, so $x^3=1$.

$$x^{2011}+\frac{1}{x^{2011}}=\left(x^3\right)^{670}\cdot x+\frac{1}{\left(x^3\right)^{670}\cdot x}=$$
$$=x+\frac{1}{x}=\frac{x^2+1}{x}=-\frac{x}{x}=-1.$$

35. (e) $\frac{m^3+1}{m-1}=m^2+m+1+\frac{2}{m-1}$
So $m-1\in\{-2,-1,1,2\}$, i.e. $m\in\{-1,0,2,3\}$. Hence the sum is 4.
36. (c)

$$\log_{35}28=\frac{\log_{14}28}{\log_{14}35}=\frac{\log_{14}14+\log_{14}2}{\log_{14}5+\log_{14}7}=$$
$$=\frac{1+\log_{14}\frac{14}{7}}{a+b}=\frac{1+\log_{14}14-\log_{14}7}{a+b}=\frac{2-a}{a+b}.$$

37. (d)

$$(1+i)^2=2i\qquad(1+i)^4=-4$$
$$(1-i)^2=-2i\qquad(1-i)^4=-4$$
$$(1+i)^{2011}+(1-i)^{2011}=\left((1+i)^4\right)^{502}\cdot(1+i)^3+\left((1-i)^4\right)^{502}\cdot(1-i)^3$$
$$=(-4)^{502}(2i-2)+(-4)^{502}\cdot(-2i-2)$$
$$=4^{502}\left(2i-2+(-2i)-2\right)=-4^{503}$$

38. (a) ${x_1}^3 - {x_2}^3 = (x_1 - x_2)({x_1}^2 + x_1x_2 + {x_2}^2)$
${x_1}^2 + x_1x_2 + {x_2}^2 = 7$.
Then $q = x_1x_2 = \frac{1}{3}\left[\left({x_1}^2 + x_1x_2 + {x_2}^2\right) - (x_1 - x_2)^2\right] = \frac{1}{3}(7 - 25) = -6$
$p^2 = (x_1 + x_2)^2 = (x_1 - x_2)^2 + 4x_1x_2 = 25 - 24 = 1$
$\Rightarrow p^2 + q = -6 + 1 = -5$

39. (b) This is really meant to solve the system of equations $x + y = 10$ and

$$\frac{.5x + .3y}{x + y} = .45$$

The result is the system of linear equations $x+y = 10$ and $.5x+.3y = 4.5$. We get $x = 7.5$ and $y = 2.5$

40. (d)

$$n + d + q = 217$$
$$n = q + 12$$
$$.10d = 4.90 + .05n$$
$$n + 49 + .5n + n - 12 = 217$$
$$n = 72, d = 85, q = 60$$