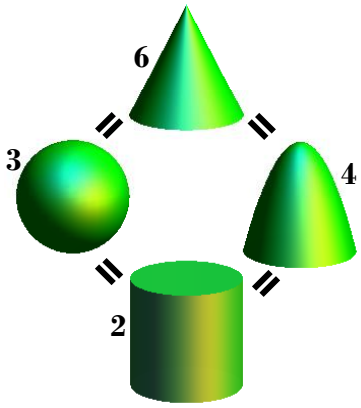
2011
High School Math Contest

Algebra II Exam



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Do's and Don'ts:

- **Do NOT** open this booklet until you are instructed to do so.
- **NO Calculators (Or other electronic devices)**
- Contestants with electronic devices (on or off) will be **disqualified!**
  And their **team** will be **disqualified!**
  (The other team members will continue to participate in the individual contest.)

DIRECTIONS:

- **The Score Card:**
  - For the ALGEBRA II exam, you must use the scorecard highlighted in **ORANGE**.
  - Write:
    - Your **NAME** on the "**name line**" (of course).
    - "**ALGEBRA II**" on the "**subject line**".
    - Your **SCHOOL** on the "**Date Line**"
- **Clearly mark** ONE bubble using **#2 PENCIL**.
  - Light marks will be counted as unmarked!
  - **Completely erase** any changes.
- You **can** write on this test booklet. (But the test booklet will not be graded.)
- **Tie Breakers**: In case of ties, the person with the least number of *wrong answers* wins. (Thus, a ***blank*** is better than ***incorrect!***)
- **The Exam:** **40** problems, **70** minutes.
- **WAIT** for the signal to begin.

1. What is the equation of the line tangent to the parabola given by $y = -3x^2 + 6x - 1$ at its vertex?
   (a) $x = 2$ (b) $y = 2$ (c) $x = 1$ (d) $y = -1$ (e) $y = 0$

2. Find all vertical asymptotes to the rational function $f(x) = \frac{2x^2-6x-20}{x^2+9x+14}$.
   (a) $y = -7$ (b) $y = -2$ and $y = -7$ (c) $x = -2$ and $x = -7$ (d) $x = -7$ (e) $x = -2$

3. Let $f(x) = x - 2$ and $g(x) = \frac{3}{x}$. What is the value of $\frac{(f\circ g)(4)}{(g\circ f)(4)}$?
   (a) $-\frac{5}{6}$ (b) $-\frac{6}{5}$ (c) $\frac{5}{6}$ (d) $\frac{6}{5}$ (e) $\frac{3}{2}$

4. How many real roots does the function $g(x) = x^4 - 16$ have?
   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

5. Let (x_0, y_0) be the point of intersection for the lines given by
$$2x + y = -1 \text{ and } -4x - 3y = 0.$$
   Calculate $3x_0 + 2y_0$.
   (a) $\frac{3}{2}$ (b) 2 (c) $-\frac{3}{2}$ (d) 3 (e) $-\frac{1}{2}$

6. The expression:
$$\frac{(-a)^3 - \left(\frac{1}{a}\right)^{-2}}{-a^2 + \left(-\frac{1}{a}\right)^{-3}}$$
   is equivalent to which of the following.
   (a) a (b) 1 (c) -1 (d) a^4 (e) $2a^{-1}$

7. If x, y, z satisfy the following system of equations, what is $x + y + z$?
$$\begin{cases} x + y - z = 3 \\ 2x - y + z = 2 \\ -x - y + 3z = 1 \end{cases}$$
   (a) 1/3 (b) -3 (c) 0 (d) 9/2 (e) 7

8. What kind of function could the following table of values represent?

x	$F(x)$
0	16
1	24
2	36
3	54
4	81

   (a) Linear function (b) Quadratic function (c) Exponential function
   (d) Logarithmic function (e) None of the answers (a) through (d) is correct.

9. $\ln\left(\frac{7x\sqrt{3-4x}}{2(x-1)^3}\right)$ can also be written as:
   (a) $\ln 7x + 2\ln(3-4x) - \ln 2 - 3\ln(x-1)$ (b) $\ln 7 + \ln x + \frac{1}{2}\ln(3-4x) - \ln 2 + 3\ln(x-1)$
   (c) $\ln 7x - \ln x - \frac{1}{2}\ln(3-4x) + \ln 2 + 3\ln(x-1)$ (d) $\ln 7 + \ln x + \frac{1}{2}\ln(3-4x) - \ln 2 - 3\ln(x-1)$
   (e) None of the answers (a) through (d) is correct.

10. Farmer Brown recently purchased a triangular piece of land that can be mapped on a coordinate grid with corners at points (-4, 2), (-1, 0), (3, 8). What is the area of this parcel given that one linear unit on the map corresponds to one mile?

    (a) 32 square miles (b) 16 square miles (c) 64 square miles
    (d) $\sqrt{16}+\sqrt{85}+\sqrt{80}$ square miles (e) None of the answers (a) through (d) is correct.

11. The endpoints of a diameter of a circle are given by (-2, -1) and (4, 5). What is the equation of the circle in standard form?

    (a) $(x-1)^2+(y-2)^2=72$ (b) $(x-2)^2+(y-4)^2=18$ (c) $(x-1)^2-(y-2)^2=72$
    (d) $(x-1)^2+(y-2)^2=18$ (e) None of the answers (a) through (d) is correct.

12. The function $f(x)=-2x^3+13x^2-17x-12$ has a root at $x=3$. Find the product of all roots of this polynomial.

    (a) -6 (b) -2 (c) 2 (d) 6 (e) 6.5

13. Let $f(x)=x^2+3$ with $x<-1$. What is the domain of the inverse function?

    (a) $(-1,4)$ (b) $(4,\infty)$ (c) $(-1,\infty)$ (d) All real numbers
    (e) None of the answers (a) through (d) is correct.

14. Which of the following equations of circles meet the criteria that they share a common point $(5,1)$ with the circle C described by the equation $(x-2)^2+(y-1)^2=9$ and also have exactly $\frac{1}{3}$ the area of circle C?

    (a) $(x-(5-\sqrt{3}))^2+(y-1)^2=3$ (b) $(x-5)^2+(y-1)^2=3$ (c) $(x-5)^2+(y-(1-\sqrt{3}))^2=9$
    (d) $x^2+22(y-1)=3$ (e) None of the answers (a) through (d) is correct.

15. Start with the graph of $y=3(x-2)^2+6$. What is the equation of the graph which results from: Shifting the graph to the left 3 units, then vertically stretching it by 5, and then shifting it up 1 unit.

    (a) $y=15(x-5)^2+7$ (b) $y=15(x+1)^2+31$ (c) $y=15(x+1)^2+7$ (d) $y=15(x-1)^2+3$
    (e) None of the answers (a) through (d) is correct.

16. Let $x=(\log_2 3)(\log_3 4)(\log_4 5)\cdot\ldots\cdot(\log_{63}64)$. Then x satisfies which of the following?

    (a) $x=6$ (b) $5<x<6$ (c) $4<x<5$ (d) $x=\log_{64}2$ (e) $x=\log_{(2\cdot 3\cdot\ldots\cdot 63)}(3\cdot 4\cdot\ldots\cdot 64)$

17. If $g(x)=2-x^2$ and $f(g(x))=\frac{1-x^2}{1+x^2}$, find $f(1)$.

    (a) 0 (b) -1 (c) 1 (d) undefined (e) None of the answers (a) through (d) is correct.

18. If $2^{2x}+16=10\cdot 2^x$, then x^2+1 is

    (a) 2 or 10 (b) 5 or 65 (c) 2 (d) 10 (e) 1

19. Let $m\neq -1$ be a real number and x_1 and x_2 the solutions of the equation

$$(x^2-6x+5)+m(x^2-5x+6)=0.$$

    Find the value of $x_1+x_2+x_1x_2$.

    (a) $\frac{m+11}{m+1}$ (b) 11 (c) $\frac{11}{m+11}$ (d) 0 (e) 1

20. Find the natural number n that satisfies the following equation:

$$\frac{(n+2)!}{(n+2)\cdot(n-1)!}=30.$$

    (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

21. At what point(s) does the line given by $x-y=-1$ intersect the circle given by $x^2+y^2-2x-2y-23=0$?
(a) $(-4,-5)$ (b) $(3,2)$ and $(-4,-5)$ (c) $(-3,-2)$ and $(4,5)$ (d) $(3,2)$ (e) $(-3,-2)$

22. Let x be the solution of x: $9^{2-5x}=27$. Find $\log x$.
(a) $\frac{1}{10}$ (b) -1 (c) 1 (d) $-\frac{1}{10}$ (e) None of the answers (a) through (d) is correct.

23. Find the inverse to the function $h(x)=4-3^{x-2}$.
(a) The inverse of the function does not exist (b) $\log_3(4-x)+2$ (c) $\log_3(x-4)+2$
(d) $\log_3(x-4)-2$ (e) $\log_3(4-x)-2$

24. Which line is perpendicular to $2x-y=4$ and passes through the vertex of the parabola $y=x^2-6x+7$?
(a) $y=2x-4$ (b) $y=\frac{1}{2}x-\frac{7}{2}$ (c) $y=2x+\frac{5}{2}$ (d) $y=-\frac{1}{2}x-\frac{1}{2}$ (e) $y=-\frac{1}{2}x+\frac{7}{2}$

25. What is the domain of the following function:

$$f(x)=\frac{3x}{\sqrt{-2x^2-9x+18}}$$

(a) $(-3,\frac{1}{2})$ (b) $(-\infty,18)$ (c) $(-6,\frac{3}{2})$ (d) $(-\infty,-6)$ and $(\frac{3}{2},\infty)$ (e) $[\frac{3}{2},18]$

26. Twelve pounds of a mixture of nuts contains \$30 worth of one kind of nut and \$70 worth of a second, more expensive kind of nut. If the difference in price of the nuts is \$4 per pound, how many more pounds of the more expensive nut are in the mixture?
(a) 3 pounds (b) 4 pounds (c) 5 pounds (d) 6 pounds
(e) None of the answers (a) through (d) is correct.

27. Which cubic polynomial below has the following roots $x=3$, $x=-5i$?
(a) $x^3-(5i+3)x^2+15ix$ (b) $x^3+(5i-3)x^2-15ix$ (c) $x^3-3x^2+25x-75$ (d) $x^3+3x^2+25x+75$
(e) None of the answers (a) through (d) is correct.

28. Steve has \$8500 to invest at simple annual interest. He will invest part at 7% and the remainder at 8%. If his goal is to collect at least \$638 interest at the end of one year, rounded to the nearest cent, what minimum amount should he invest at 8%?
(a) \$4200 (b) \$4300 (c) \$4253.33 (d) \$4246.67
(e) None of the answers (a) through (d) is correct.

29. In a solar system outside of our galaxy, a planet following the orbit $x^2+y^2-64=0$ around a sun located at the origin is approached by a comet moving along the path defined by $x^2-5y-3=0$ in the same plane as the planet's orbit. Calculate where the comet's path will cross the planet's orbit.
(a) $\left(\sqrt{5\frac{\sqrt{269}-5}{2}+3},\frac{\sqrt{269}-5}{2}\right)$ (b) $\left(-\sqrt{5\frac{\sqrt{269}-5}{2}+3},\frac{\sqrt{269}-5}{2}\right)$ (c) Both (a) and (b)
(d) No real solution (e) None of the answers (a) through (d) is correct.

30. Which of the following is not a vertex of the system:

$$\begin{aligned} x-8y &\geq -8 \\ 5x+3y &\geq 3 \\ 6x-5y &\leq 3 \end{aligned}$$

(a) $(0,1)$ (b) $\left(\frac{64}{43},1\frac{8}{43}\right)$ (c) $\left(\frac{3}{5},0\right)$ (d) $\left(\frac{24}{43},\frac{3}{43}\right)$
(e) None of the answers (a) through (d) is correct.

31. A colony of bacteria living on a Petri dish under optimal conditions doubles in size every 10 minutes. At noon a certain day, the Petri dish is completely covered with bacteria. At what time was 25% of the plate covered by bacteria?

(a) 10:45 am (b) 11:15 am (c) 3 am (d) 11:40 am
(e) None of the answers (a) through (d) is correct.

32. If $(2x-1)^{2011} = a_0 + a_1x + a_2x^2 + \ldots + a_{2011}x^{2011}$, then the value of $a_0 + a_1 + a_2 + \ldots + a_{2011}$ is:

(a) 1 (b) between 2^{2010} and 2^{2011} (c) -1 (d) 0
(e) None of the answers (a) through (d) is correct.

33. For how many values of the real number a do the equations

$$x^2 + ax + 1 = 0$$
$$x^2 - x - a = 0$$

have a common real solution?

(a) 0 (b) 1 (c) 2 (d) 3 (e) An infinite number of values

34. Let x be a solution to $x^2 + x + 1 = 0$. Find the value of $x^{2011} + \frac{1}{x^{2011}}$.

(a) $\left(\frac{\sqrt{-3}}{2}\right)^{2011}$ (b) -1 (c) $\left(\frac{\sqrt{-3}}{2}\right)^{2011} + \left(\frac{2}{\sqrt{-3}}\right)^{2011}$ (d) 1 (e) 0

35. Find the sum of all integer numbers m for which $\frac{m^3+1}{m-1}$ is an integer number.

(a) 1 (b) 2 (c) -1 (d) 5 (e) 4

36. If $\log_{14} 7 = a$ and $\log_{14} 5 = b$, find $\log_{35} 28$.

(a) $\frac{a-2}{a+b}$ (b) $\frac{a+b}{2-a}$ (c) $\frac{2-a}{a+b}$ (d) $\frac{a+2}{a-b}$ (e) $\frac{a-b}{a+2}$

37. If $i^2 = -1$, find $(1+i)^{2011} + (1-i)^{2011}$.

(a) 2 (b) 0 (c) $2 + i^{2011} + (-i)^{2011}$ (d) $(-4)^{503}$ (e) 2^{503}

38. Let x_1 and x_2 be the roots of the equation $x^2 + px + q = 0$ where p and q are real numbers. Find $p^2 + q$ if $x_1 - x_2 = 5$ and ${x_1}^3 - {x_2}^3 = 35$.

(a) -5 (b) -17 (c) 7 (d) 0 (e) None of the answers (a) through (d) is correct.

39. Solution A is 50% antifreeze and 50% water. Solution B is 30% antifreeze and 70% water. You wish to mix x liters of Solution A with y liters of Solution B to form 10 liters of a new solution which is 45% antifreeze and 55% water. What is the ratio of x to y?

(a) 3:2 (b) 3:1 (c) 2:1 (d) 5:3 (e) 1:1

40. A large coin jar is full of nickels, dimes and quarters. There are 217 coins in all. The number of nickels is 12 more than the number of quarters. The value of the dimes is \$4.90 more than the value of the nickels. How many coins of each type are there in the bank?

(a) 30 nickels, 169 dimes, 18 quarters (b) 87 nickels, 55 dimes, 75 quarters
(c) 30 nickels, 145 dimes, 42 quarters (d) 72 nickels, 85 dimes, 60 quarters
(e) None of the answers (a) through (d) is correct.